Mckay corespondence in Quasitoric orbifolds

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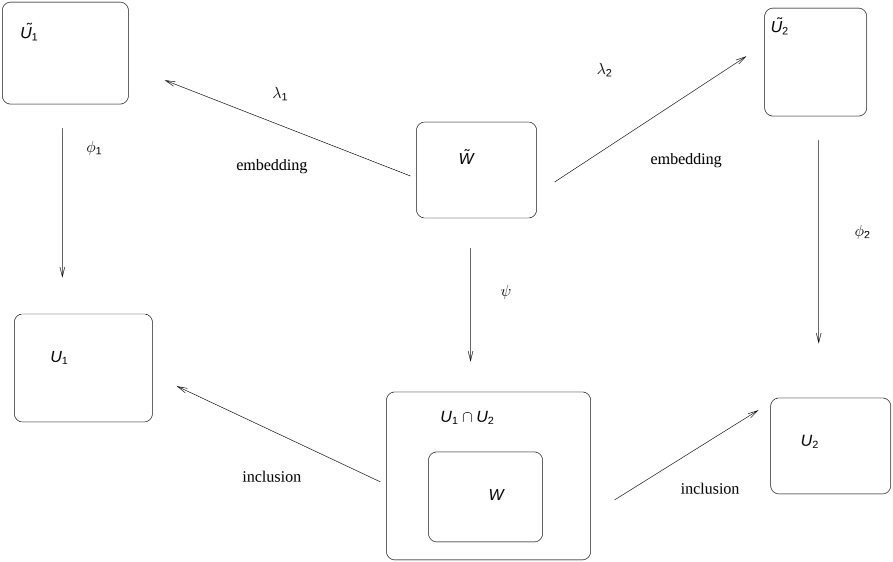
Definition

An orbifold X is a topological space locally homeomorphic to $\tilde{U}/G$ where $\tilde{U}$ is a an open connected set in R^n and G is a finiite group with a smooth action. Orbifolds have atlases and compatibility conditions similar to manifolds. For the notion of orbifold structure, not only the topology of X but the data of group actions is also important.

Orbifold structure

An orbifold structure on a topological space X is given by an atlas of uniformising charts $\{(U, \tilde{U}, G, \phi)\}$ where U is an open set in X, $\tilde{U}$ is an open connected subset in R^n, G acts smoothly on $\tilde{U}$ and $\phi : \tilde{U} \to U$ is a continuous map that induces a homeomorphism $\tilde{U}/G \to U$.

Compatibility of charts



Tangent bundle of a orbifold

Definition

The tangent bundle TU of U is defined to be the orbifold $T\tilde{U}/G$ where an element $g \in G$ acts on $T\tilde{U}$ by its differential dg.
TX is an orbifold with atlas $(TU, T\tilde{U}, G, \psi)$.

Almost Complex Structure

Definition

Let TX be the tangent bundle of the orbifold X.
An almost complex structure J is a base preserving continuous map $TX \to TX$ satisfying the following:

1. *It lifts to a fiberwise linear map $J : T\tilde{U} \to T\tilde{U}$ for every chart $(U, \tilde{U}, G, \phi)$ in the atlas*
2. $J^2 \mid (fiber) = -I$
3. *For every $g \in G$ we have $dgJdg^{-1} = J$*
4. *It respects embedding of charts.*

Definition

Let X be an almost complex orbifold. For a chart $(U, \tilde{U}, G, \phi)$ consider $\bigwedge^n (T^{1,0}(\tilde{U})^*)$ with G action

$$g(dv_1 \wedge \ldots \wedge dv_n) = g^{-1*}(dv_1 \wedge \ldots \wedge dv_n)$$

. The canonical bundle K_X is an orbifold with atlas $(|\bigwedge^n (T^{1,0}(\tilde{U})^*)|, \bigwedge^n (T^{1,0}(\tilde{U})^*), G, \psi_1)$.

Blowdown in complex orbifolds

Definition

Let Y, X be complex orbifolds, and $f : Y \to X$ a holomorphic map. The map f is called a blowdown if there exists an analytic subset X^a of X such that

$$f : Y - f^{-1}(X^a) \longrightarrow X - X^a$$

is a biholomorphism, and f^{-1} is meromorphic with singularities along X^a.

Crepant blowdown in complex orbifolds

Definition

A blow down is called f is called crepant if $f^*(K_X) = K_Y$

Projective toric orbifold

Definition

A projective toric orbifold X is a simplicial projective toric variety.Since it is simplicial it has orbifold singularity. It has a C^{*n} action with a dense principal orbit. The C^{*n} action induces a $U(1)^n$ action.

Projective toric orbifold

Definition

The quotient of the $U(1)^n$ action is a n-dimensional simple polytope. The quotient map is called π and the polytope P.

An n-dimensional polytope is simple if exactly n codimension one faces intersect at each vertex.

Projective toric orbifold

Definition

Let F be a codimension one face of P. Then $\pi^{-1}(F^{\circ})$ has a circle subgroup as its stationary subgroup which is characterized by an integral vector $\lambda(F)$, well defined up to sign (in a suitable integral basis). This integral vector $\pm\lambda(F)$ is normal to F. We call a codimension one face F a facet and $\lambda(F)$ the characteristic vector associated to the facet. We have a map λ from facets to Z^n.

Batyrev and Dais proved that string theoretic Hodge numbers are preserved for crepant blowdowns of Gorenstein complex algebraic varieties with toroidal quotient singularities. In general, McKay correspondence for arbitrary compact complex algebraic orbifolds was proved by Lupercio-Poddar and Yasuda.

In easy language quotient singularities are orbifold singularities and Gorenstein means that each dg belongs to $SL(n, C)$. From now on we call such orbifolds SL.

Since projective toric orbifolds have pure Hodge structure the above amounts to Hodge numbers and Betti numbers of Chen-Ruan cohomology being preserved under crepant blowdowns.

The proof requires manipulation of something called E-polynomial which is defined as follows.

$$e^{p,q}(X) = \Sigma_{k\geq 0}(-1)^k h^{p,q}(H_c^k(X)). \tag{1}$$

The E-polynomial is

$$E(X; u, v) := \Sigma_{p,q} e^{p,q}(X) u^p v^q. \tag{2}$$

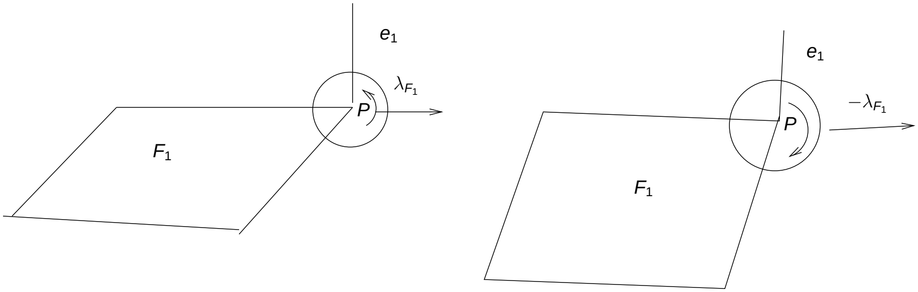
The involvement of Hodge structure restricts the domain of the result to the complex analytic category.

Definition

Quasitoric orbifolds were defined by Davis-Januswkiewicz and Poddar-Sarkar by generalizing the projective toric orbifolds. They relaxed the condition that the characteristic vectors have to be normal to the facets. They require the condition that the characteristic vectors of the facets meeting at a vertex have to be linearly independent.

Omniorientation

1. Recall: The circle subgroup generated by a characteristic vector fixes the suborbifold represented by the facet.
2. The sign of the characteristic vector defines an orientation of the normal bundle of the suborbifold. These are called omniorientations defined by the characteristic vector.
3. Given a choice of signs of characteristic vector we get an omnioriented Quasitoric orbifold.


e_1
λ_{F_1}
P
F_1
e_1
$-\lambda_{F_1}$
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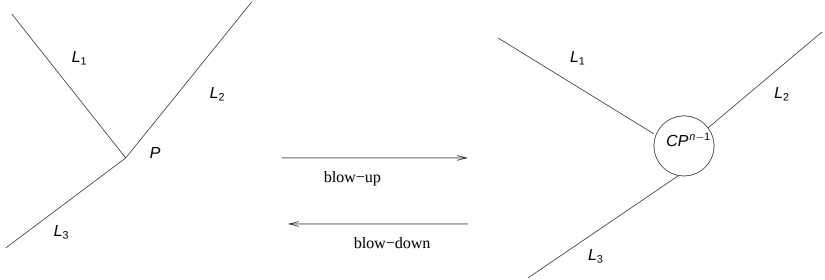
F_1

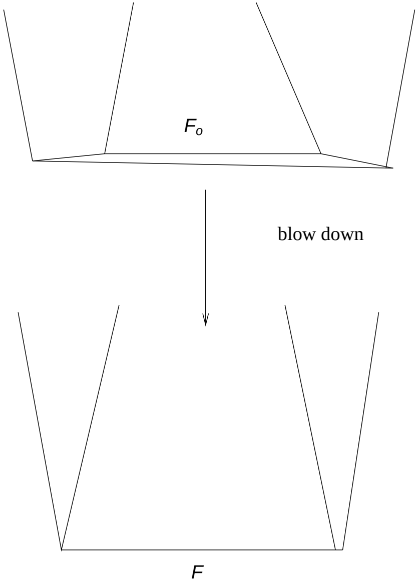
point blowup in 2n-manifold



2n MANIFOLD POINT BLOW UP

blowup and blowdown in Quasitoric Orbifolds

Topologically in a quasitoric manifold blowup will correspond to replacing an torus invariant submanifold of X by the projectivization of its normal bundle. In orbifolds we give a combinatorial description. Combinatorially we replace a face by a facet with a new characteristic vector. Let Y be the space obtained by this operation. Blowdown is the natural map $\rho : Y \to X$.



F_o
blow down
F

Let F be the blowed up codimension k face. If it has a characteristic set $\lambda_F = \{\lambda_1, \lambda_2 \ldots \lambda_k\}$. Let F_0 be the new facet after blow up. The characteristic vector λ_0 is given by

$$\lambda_0 = \sum_{j=1}^{k} b_j \lambda_j, \; b_j \in Q \cap (0,1) \, \forall j. \tag{3}$$

Crepancy

Definition

A blow up or blowdown is crepant if

$$\sum_{j=1}^{k} b_j = 1 \tag{4}$$

For almost complex quasitoric orbifolds with some more conditions the two definitions are same

Chen- Ruan cohomology

Definition

Let F be any codimension k face and λ_F be its characteristic set and $N(F)$ be the sub module of Z^n generated by the the vectors of λ_F. Define

$$G_F = \sum_{j=1}^{k} a_j \lambda_j \cap Z^n / N(F) \tag{5}$$

Here $a_j \in Q \cap [0, 1)$. G_F is a group.

For $g \in G_F$, define the degree shifting number or age of g to be

$$\iota(g) = \sum a_j. \tag{6}$$

Chen-Ruan cohomology

We write $F \leq H$ if F is a sub-face of H, and $F < H$ if it is a proper sub-face. If $F \leq H$ we have a natural inclusion of G_H into G_F induced by the inclusion of $N(H)$ into $N(F)$. Therefore we may regard G_H as a subgroup of G_F.

Define the set

$$G_F^{\circ} = G_F - \bigcup_{F<H} G_H \tag{7}$$

Chen-Ruan cohomology

We define the Chen-Ruan orbifold cohomology of an omnioriented quasitoric orbifold **X** to be

$$H^*_{CR}(\mathbf{X}, R) = \bigoplus_{F \leq P} \bigoplus_{g \in G_F^\circ} H^{*-2\iota(g)}(X(F), R).$$

An almost complex orbifold is SL if each dg is in $SL(n, C)$. This is equivalent to $\iota(g)$ being integral for every g in G_F.

Therefore, to suit our purposes, we make the following definition.

Definition

A quasitoric orbifold is said to be quasi-SL if the age of every g in G_F $\iota(g)$ is an integer.

main theorem

Theorem

For any crepant blowdown $\rho : Y \to X$ quasi-SL orbifolds, Betti numbers of Chen-Ruan cohomology of X and Y are the same.

idea of Proof

1. Singularities of a quasitoric orbifold are same as that of a projective toric orbifold. In quasitoric orbifold things don't patch analytically: So obstructions are global.
2. To handle global obstructions we use the fact that any simple polytope has combinatorially equivalent rational or integral realization.
3. From integral polytope we get projective toric orbifolds where the result holds.
4. Since singular or de Rham cohomology of of quasitoric orbifold depend on the combinatorial class of the polytope and since Chen-Ruan cohomology depends on singular cohomology of quasitoric suborbifolds, we can adapt Batyrev-Dais style of proof to our situation.