The Cohomology Algebra of Polyhedral Product Spaces

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The cohomology group

For two graded groups $A^*_\Lambda = \bigoplus_{\alpha\in\Lambda} A^*_\alpha$ and $B^*_\Lambda = \bigoplus_{\alpha\in\Lambda} B^*_\alpha$, their diagonal tensor product group with respect to the index set Λ is

$$A^*_\Lambda \otimes_\Lambda B^*_\Lambda = \bigoplus_{\alpha\in\Lambda} A^*_\alpha \otimes B^*_\alpha.$$

The usual tensor product is

$$A^*_\Lambda \otimes B^*_\Lambda = \bigoplus_{\alpha,\beta\in\Lambda} A^*_\alpha \otimes B^*_\beta.$$

The cohomology group

For a simplicial complex K with vertex set a subset of $[m]$ and a sequence of CW-complex pairs $(\underline{X}, \underline{A}) = \{(X_k, A_k)\}_{k=1}^m$, the polyhedral product space $\mathcal{Z}(K; \underline{X}, \underline{A})$ is the subspace of $X_1 \times \cdots \times X_m$ defined as follows. For a subset σ of $[m]$, define

$$D(\sigma) = Y_1 \times \cdots \times Y_m, \quad Y_k = \begin{cases} X_k & \text{if } k \in \sigma, \\ A_k & \text{if } k \notin \sigma. \end{cases}$$

Then $\mathcal{Z}(K; \underline{X}, \underline{A}) = \cup_{\sigma \in K} D(\sigma)$.

The cohomology group

Let $\mathcal{Z}(K;\underline{X},\underline{A})$ be a polyhedral product space such that every $\ker i_k^*$, $\operatorname{coker} i_k^*$, $\operatorname{im} i_k^*$ are free modules over $\mathbb{F}$, where

$$i_k^*\colon H^*(X_k;\mathbb{F}) \to H^*(A_k;\mathbb{F})$$

is the singular cohomology homomorphism induced by the inclusion map. The cohomology group of the polyhedral product space is

$$H^*(\mathcal{Z}(K;\underline{X},\underline{A});\mathbb{F}) \cong H^*_{\Upsilon_m}(K) \otimes_{\Upsilon_m} H^*_{\Upsilon_m}(\underline{X},\underline{A}).$$

The cohomology group

The index set $\Upsilon_m = \{(\sigma, \omega) \,|\, \sigma, \omega \subset [m],\, \sigma \cap \omega = \emptyset\}$.

The total cohomology group of K over $\mathbb{F}$ is

$$H^*_{\Upsilon_m}(K) = \oplus_{(\sigma,\,\omega)\in\Upsilon_m} H^*_{\sigma,\,\omega}(K),$$

where $H^*_{\sigma,\,\omega}(K) = \widetilde{H}^{*-1}(K_{\sigma,\,\omega};\mathbb{F})$, the singular cohomology with degree uplifted by 1 and

$$K_{\sigma,\,\omega} = \mathrm{link}_K\sigma|_\omega = \{\tau \,|\, \tau\subset\omega,\, \sigma\cup\tau \in K,\, \sigma\cap\tau = \emptyset\}$$

if $\sigma \in K$ and $K_{\sigma,\,\omega} = \{\}$ if $\sigma \notin K$.

The cohomology group

The cohomology group of $(\underline{X}, \underline{A})$ over $\mathbb{F}$ is

$$H^*_{\Upsilon_m}(\underline{X}, \underline{A}) = \bigoplus_{(\sigma, \omega)\in\Upsilon_m} H^*_{\sigma, \omega}(\underline{X}, \underline{A}),$$

where $H^*_{\sigma, \omega}(\underline{X}, \underline{A}) = H^1 \otimes \cdots \otimes H^m$ with

$$H^k = \begin{cases} \ker i_k^* & \text{if } k \in \sigma, \\ \operatorname{coker} i_k^* & \text{if } k \in \omega, \\ \operatorname{im} i_k^* \cong \operatorname{coim} i_k^* & \text{otherwise.} \end{cases}$$

The cohomology algebra

An algebra (A^*, Π) is a graded group A^* with product

$$\Pi\colon A^* \otimes A^* \to A^*$$

a graded group homomorphism. Π may not be associative.

If an algebra (A^*_Λ, Π) satisfies $A^*_\Lambda = \oplus_{\alpha\in\Lambda} A^*_\alpha$, then the product Π is determined by all its restriction product

$$\Pi_\alpha^{\beta,\gamma}\colon A^*_\beta \otimes A^*_\gamma \xrightarrow{i} A^*_\Lambda \otimes A^*_\Lambda \xrightarrow{\Pi} A^*_\Lambda \xrightarrow{p} A^*_\alpha,$$

where i is the inclusion and p is the projection, since for $b \in A^*_\beta$ and $c \in A^*_\gamma$,

$$\Pi(b\otimes c) = \Sigma_{\alpha\in\Lambda} \Pi_\alpha^{\beta,\gamma}(b\otimes c) \text{ with } \Pi_\alpha^{\beta,\gamma}(b\otimes c) \in A^*_\alpha.$$

The cohomology algebra

For algebras $(A^*_\Lambda = \oplus_{\alpha\in\Lambda} A_\alpha, \Pi_1)$ and $(B^*_\Lambda = \oplus_{\alpha\in\Lambda} B_\alpha, \Pi_2)$, their diagonal tensor product algebra $(A^*_\Lambda \otimes_\Lambda B^*_\Lambda, \Pi_1 \otimes_\Lambda \Pi_2)$ with respect to Λ is defined as follows. For

$$a'_\beta \in A^*_\beta,\ a''_\gamma \in A^*_\gamma \text{ with } \Pi_1(a'_\beta \otimes a''_\gamma) = \Sigma_{\alpha\in\Lambda}\, a_\alpha,\ a_\alpha \in A^*_\alpha,$$

$$b'_\beta \in B^*_\beta,\ \ b''_\gamma \in B^*_\gamma \text{ with } \ \Pi_2(b'_\beta \otimes b''_\gamma) = \Sigma_{\alpha\in\Lambda} b_\alpha,\ \ b_\alpha \in B^*_\alpha,$$

$$(\Pi_1 \otimes_\Lambda \Pi_2)\Big((a'_\beta \otimes b'_\beta) \otimes (a''_\gamma \otimes b''_\gamma)\Big) = (-1)^{|a''_\gamma||b'_\beta|} \Sigma_{\alpha\in\Lambda}\, a_\alpha \otimes b_\alpha.$$

Equivalently, the restriction products of the three algebras satisfy

$$(\Pi_1 \otimes_\Lambda \Pi_2)^{\beta,\gamma}_\alpha = (\Pi_1)^{\beta,\gamma}_\alpha \otimes (\Pi_2)^{\beta,\gamma}_\alpha.$$

The cohomology algebra

Let $\mathcal{Z}(K; \underline{X}, \underline{A})$ be a polyhedral product space such that every $\ker i_k^*$, $\operatorname{coker} i_k^*$, $\operatorname{im} i_k^*$ are free modules over $\mathbb{F}$, the cohomology algebra of the polyhedral product space is

$$
\begin{aligned}
&(H^*(\mathcal{Z}(K; \underline{X}, \underline{A}); \mathbb{F}), \cup) \\
\cong\ &(H^*_{\Upsilon_m}(K) \otimes_{\Upsilon_m} H^*_{\Upsilon_m}(\underline{X}, \underline{A}),\ \cup_K \otimes_{\Upsilon_m} \Pi_{(\underline{X}, \underline{A})}).
\end{aligned}
$$

The cohomology algebra

The universal cohomology algebra $(H^*_{\Upsilon_m}(K), \cup_K)$ is defined as follows. The restriction product

$$\cup_K^R \colon H^*_{\sigma', \omega'}(K) \otimes H^*_{\sigma'', \omega''}(K) \to H^*_{\sigma, \omega}(K)$$

of $\cup_K$ is induced by the cochain complex homomorphism

$$\Pi_K^R \colon \widetilde{C}^*(K_{\sigma', \omega'}) \otimes \widetilde{C}^*(K_{\sigma'', \omega''}) \to \widetilde{C}^*(K_{\sigma, \omega})$$

defined as follows.

The cohomology algebra

(1) $\Pi_K^R = 0$ if $\sigma'\cup\sigma'' \not\subset \sigma$ or $\omega \not\subset \omega'\cup\omega''$.
(2) For $\{i_1, \cdots, i_s\} \in \widetilde{C}^*(K_{\sigma',\omega'})$, $\{j_1, \cdots, j_t\} \in \widetilde{C}^*(K_{\sigma'',\omega''})$ and $\sigma'\cup\sigma'' \subset \sigma$, $\omega \subset \omega'\cup\omega''$,

$$\Pi_K^R(\{i_1, \cdots, i_s\}\otimes\{j_1, \cdots, j_t\}) = 0$$

if $\{i_1, \cdots, i_s\} \cap \{j_1, \cdots, j_t\} \neq \emptyset$ or $\{j_1, \cdots, j_t\} \cap \omega' \neq \emptyset$, and otherwise,

$$\Pi_K^R(\{i_1, \cdots, i_s\}\otimes\{j_1, \cdots, j_t\}) = (-1)^\tau\{k_1, \cdots, k_u\},$$

where $\{k_1, \cdots, k_u\} = \{i_1, \cdots, i_s\} \cup \{j_1, \cdots, j_t\}$ and $(-1)^\tau$ is the sign of the permutation $\begin{pmatrix} i_1 & \cdots & i_s & j_1 & \cdots & j_t \\ k_1 & \cdots & k_s & k_{s+1} & \cdots & k_u \end{pmatrix}$ and s, t, u may be 0.

The cohomology algebra

The cohomology algebra of $(\underline{X}, \underline{A})$ is

$$(H^*_{\Upsilon_m}(\underline{X}, \underline{A}), \Pi_{(\underline{X}, \underline{A})})$$

$$= (H^*_{\Upsilon}(X_1, A_1) \otimes \cdots \otimes H^*_{\Upsilon}(X_m, A_m), \Pi_1 \otimes \cdots \otimes \Pi_m)$$

with Π_k on $H^*_{\Upsilon}(X_k, A_k) = \ker i^*_k \oplus \operatorname{coker} i^*_k \oplus \operatorname{im} i^*_k$ defined

as follows.

(1) $\Pi_k(x \otimes y) = \Pi_{X_k}(x \otimes y)$ for all
$x, y \in \ker i^*_k \oplus \operatorname{im} i^*_k = \ker i^*_k \oplus \operatorname{coim} i^*_k = H^*(X_k)$,
(2) $\Pi_k(a \otimes b) = \Pi_{A_k}(a \otimes b)$ for all
$a, b \in H^*(A_k)$ such that a or b is in $\operatorname{coker} i^*_k$,
(3) $\Pi_k(x \otimes a) = \Pi_k(a \otimes x) = 0$ for all $x \in \ker i^*_k$ and $a \in \operatorname{coker} i^*_k$.

The applications

The normal algebra $(H^*_{\Upsilon_m}(K), \widetilde{\cup}_K)$ of K is defined as follows. The restriction product

$$H^*_{\sigma', \omega'}(K) \otimes H^*_{\sigma'', \omega''}(K) \to H^*_{\sigma, \omega}(K)$$

of $\widetilde{\cup}_K$ coincides with the restriction product of the universal product $\cup_K$ if $\sigma' \cup \sigma'' = \sigma$ and $\omega = \omega' \cup \omega''$ and all other restriction products of $\widetilde{\cup}_K$ is 0.

The special algebra $(H^*_{\Upsilon_m}(K), \overline{\cup}_K)$ of K is defined as follows. The restriction product

$$H^*_{\sigma', \omega'}(K) \otimes H^*_{\sigma'', \omega''}(K) \to H^*_{\sigma, \omega}(K)$$

of $\overline{\cup}_K$ coincides with the restriction product of the universal product $\cup_K$ if $\sigma' \sqcup \sigma'' = \sigma$ and $\omega = \omega' \sqcup \omega''$ and all other restriction products of $\widetilde{\cup}_K$ is 0.

The applications

Suppose for $k = 1, \cdots, m$, $H^*_{\Upsilon}(X_k, A_k)$ is a free module such that $\operatorname{coker} i_k^*$ is an ideal of $H^*(A_k)$ and $\operatorname{coim} i_k^* = \operatorname{im} i_k^*$ is a subalgebra of both $H^*(A_k)$ and $H^*(X_k)$. So $(H^*_{\Upsilon_m}(\underline{X}, \underline{A}), \Pi_{(\underline{X}, \underline{A})})$ is an associative, commutative algebra with unit. Then

$$(H^*(\mathcal{Z}(K; \underline{X}, \underline{A})), \cup)$$

$$\cong (H^*_{\Upsilon_m}(K) \otimes_{\Upsilon_m} H^*_{\Upsilon_m}(\underline{X}, \underline{A}), \widetilde{\cup}_K \otimes_{\Upsilon_m} \Pi_{(\underline{X}, \underline{A})}).$$

Specifically,

$$(H^*(\mathcal{Z}(K; S\underline{X}, S\underline{A})), \cup)$$

$$\cong (H^*_{\Upsilon_m}(K) \otimes_{\Upsilon_m} H^*_{\Upsilon_m}(S\underline{X}, S\underline{A}), \overline{\cup}_K \otimes_{\Upsilon_m} \Pi_{(\underline{X}, \underline{A})}).$$

The applications

Let $\mathcal{L}_m = \{(\sigma, \emptyset) \in \Upsilon_m\}$, $\mathcal{R}_m = \{(\emptyset, \omega) \in \Upsilon_m\}$.

The left universal algebra $(H^*_{\mathcal{L}_m}(K), \cup_K)$ of K is the subalgebra of $(H^*_{\Upsilon_m}(K), \cup_K)$ with $H^*_{\mathcal{L}_m}(K) = \oplus_{(\sigma,\emptyset)\in\mathcal{L}_m} H^*_{\sigma,\emptyset}(K)$.

The right universal algebra $(H^*_{\mathcal{R}_m}(K), \cup_K)$ of K is the quotient algebra of $(H^*_{\Upsilon_m}(K), \cup_K)$ over the ideal $\oplus_{(\sigma,\omega)\in\Upsilon_m,\,\sigma\neq\emptyset} H^*_{\sigma,\omega}(K)$.

When the vertex set of K is $[m]$, by Hochster theorem,

$$H^*_{\mathcal{R}_m}(K) \cong \oplus_{\omega\subset[m]} H^*(K|_\omega) \cong \mathrm{Tor}^*_{\mathbb{F}[x_1,\cdots,x_m]}(\mathbb{F}(K), \mathbb{F}),$$

where $\mathbb{F}(K)$ is the Stanley-Reisner face ring of K.

The applications

Suppose $H^*_\Upsilon(X_k, A_k)$ is a free module.

The left cohomology algebra $H^*_{\mathcal{L}_m}(\underline{X}, \underline{A})$ of $(\underline{X}, \underline{A})$ is the subalgebra of $H^*_{\Upsilon_m}(\underline{X}, \underline{A})$ with

$$H^*_{\mathcal{L}_m}(\underline{X}, \underline{A}) = \oplus_{(\sigma,\emptyset)\in\mathcal{L}_m} H^*_{\sigma,\emptyset}(\underline{X}, \underline{A}).$$

The right cohomology algebra $H^*_{\mathcal{R}_m}(\underline{X}, \underline{A})$ of $(\underline{X}, \underline{A})$ is the quotient algebra of $H^*_{\Upsilon_m}(\underline{X}, \underline{A})$ modular the ideal

$$\bigoplus_{(\sigma,\omega)\in\Upsilon_m,\sigma\neq\emptyset} H^*_{\sigma,\omega}(\underline{X}, \underline{A}).$$

The applications

If every $i_k^*\colon H^*(X_k) \to H^*(A_k)$ is an epimorphism between free modules, then

$$\Big(H^*(\mathcal{Z}(K;\underline{X},\underline{A})),\cup\Big) \cong \Big(H^*_{\mathcal{L}_m}(\underline{X},\underline{A}),\ \Pi_{(\underline{X},\underline{A})}\Big).$$

Specifically, if $(X_k, A_k) = (CP^\infty, *)$ for all k, then

$$H^*_{\mathcal{L}_m}(\underline{X},\underline{A}) \cong \mathbb{F}(K).$$

If every $i_k^*\colon H^*(X_k) \to H^*(A_k)$ is a monomorphism between free modules, then

$$\Big(H^*(\mathcal{Z}(K;\underline{X},\underline{A})),\cup\Big)$$

$$\cong \Big(H^*_{\mathcal{R}_m}(K)\otimes_{\mathcal{R}_m} H^*(A_1\times\cdots\times A_m),\ \cup_K\otimes_{\mathcal{R}_m}\cup\Big).$$