Simple polytopes and simplicial complexes with Buchstaber number two

Nikolay Erokhovets

Lomonosov Moscow State University

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Toric topology

Canonical correspondence

simplicial complex K		moment-angle complex $\mathcal{Z}_K$
$\dim K = n-1$	$\longrightarrow$	$\dim \mathcal{Z}_K = m+n$
number of vertices $= m$		canonical T^m-action
Combinatorics of K	$\longleftrightarrow$	Topology of $\mathcal{Z}_K$

K-power

$A \subset X$ – a pair of topological spaces, $[m] = \{1, \ldots, m\}$.

$$(X, A)^K = \bigcup_{\sigma \in K} X^\sigma \times A^{[m]\setminus\sigma} \subset X^m,$$

where $X^\sigma \times A^{[m]\setminus\sigma} = X_1 \times \cdots \times X_m$, $X_i = \begin{cases} X, & i \in \sigma \\ A, & i \in [m] \setminus \sigma \end{cases}$

Example

$D^2 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| \leqslant 1\}$, $S^1 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| = 1\}$,

$(D^2, S^1)^K$ – a moment-angle complex $\mathcal{Z}_K$.

$D^1 = \{x \in \mathbb{R} \colon |x| \leqslant 1\}$, $S^0 = \{\pm 1\}$,

$(D^1, S^0)^K$ – a real moment-angle complex $\mathbb{R}\mathcal{Z}_K \subset \mathcal{Z}_K$.

There are canonical coordinate actions of $T^m = (S^1)^m$ on $\mathcal{Z}_K$, and $(S^0)^m \simeq \mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$.

Definition

A Buchstaber invariant $s(K)$ is the maximal dimension r of toric subgroups $H \subset T^m$, $H \simeq T^r$, that act freely on $\mathcal{Z}_K$.

Definition

A real Buchstaber invariant $s_{\mathbb{R}}(K)$ is the maximal dimension r of subgroups $H_2 \subset \mathbb{Z}_2^m$ that act freely on $\mathbb{R}\mathcal{Z}_K$.

$$s(\Delta^{n-1}) = s_{\mathbb{R}}(\Delta^{n-1}) = 0,$$

$$1 \leqslant s(K) \leqslant s_{\mathbb{R}}(K) \leqslant m - n, \qquad K \neq \Delta^{n-1}$$

Buchstaber problem

Problem (V. M. Buchstaber, 02)

To find an EFFECTIVE combinatorial description of $s(K)$.

Modifications (V. M. Buchstaber, 12)

Problem’

For any n to calculate $s(K)$ for all simplicial complexes with $\dim K = n-1$.

Problem”

For any r to characterize combinatorially simplicial complexes K with $s(K) = r$.

Two descriptions of a toric subgroup $\Rightarrow$

(S) $s(K)$ is the maximal r that admits a matrix $S \in \mathbb{Z}^{m\times r}$ such that for any $\sigma \in K$, the rows $\{S^i : i \in [m] \setminus \sigma\}$ span $\mathbb{Z}^r$;

(Λ) $s(K)$ is the maximal r that admits a mapping $\Lambda \colon [m] \to \mathbb{Z}^{m-r}$ such that for any simplex $\sigma \in K$ the vectors $\Lambda(\sigma)$ form part of a basis in $\mathbb{Z}^{m-r}$.

Lemma

1. $s(K) \geqslant r \iff s_{\mathbb{R}}(K) \geqslant r$ *for* $r = 1, 2, 3$.
2. $s(K) = s_{\mathbb{R}}(K)$ *for* $\dim K = 0, 1, 2$.

Corollary

$s(K) = 2$ *if and only if* $s_{\mathbb{R}}(K) = 2$.

Generalized chromatic number

Definition

Chromatic number $\gamma(G)$ of a graph $G = (V, E)$ is a minimal r that allows a coloring $\Lambda\colon V \to [r] = \{1, \ldots, r\}$ such that

$$\{i, j\} \in E \Rightarrow \Lambda(i) \neq \Lambda(j).$$

$s(K) =$ maximal r that allows a coloring $\Lambda\colon \mathrm{Vert}(K) \to \mathbb{Z}^{m-r}$ such that

$$\{i_1, \ldots, i_l\} \in K \Rightarrow \{\Lambda(i_1), \ldots, \Lambda(i_l)\} \text{ – part of a basis in } \mathbb{Z}^{m-r}.$$

Missing simplices

Definition

The set $\omega \subset [m]$ is called a missing simplex, if $\omega \notin K$ and any proper subset belongs to K.

Denote by $N(K)$ the set of all missing simplices.

$$\sigma \in K \Leftrightarrow \nexists \omega \in N(K) : \omega \subset \sigma$$

Thus, $N(K)$ determines K in a unique way.

$$K = \Delta^n \Leftrightarrow N(K) = \varnothing;$$

Problem": Simplicial complexes

Proposition

$s(K) \geqslant 1$ *if and only if* $N(K) \neq \varnothing$, *i.e.* $K \neq \Delta^n$.

Proposition (E.,13)

$s(K) \geqslant 2$ *if and only if* $N(K)$ *contains one of the subsets:*

1. $\{\tau_1, \tau_2, \tau_3\}$: $\tau_1 \cap \tau_2 \cap \tau_3 = \varnothing$;
2. $\{\tau_1, \tau_2\}$: $\tau_1 \cap \tau_2 = \varnothing$.

Theorem (E.,13)

$s(K) \geqslant 3$ if and only if $N(K)$ contains one of the subsets:

1. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7\}$: $\tau_1 \cap \tau_2 \cap \tau_4 = \varnothing$; $\tau_1 \cap \tau_3 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_6 \cap \tau_7 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_6 = \varnothing$; $\tau_2 \cap \tau_5 \cap \tau_7 = \varnothing$; $\tau_3 \cap \tau_4 \cap \tau_7 = \varnothing$; $\tau_4 \cap \tau_5 \cap \tau_6 = \varnothing$;
2. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6\}$: $\tau_1 \cap \tau_3 = \varnothing$; $\tau_1 \cap \tau_2 \cap \tau_4 = \varnothing$; $\tau_1 \cap \tau_2 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_4 \cap \tau_6 = \varnothing$; $\tau_1 \cap \tau_5 \cap \tau_6 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_6 = \varnothing$; $\tau_3 \cap \tau_4 \cap \tau_5 = \varnothing$;
3. $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$: $\tau_1 \cap \tau_2 = \varnothing$; $\tau_1 \cap \tau_5 = \varnothing$; $\tau_1 \cap \tau_3 \cap \tau_4 = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_5 = \varnothing$; $\tau_2 \cap \tau_4 \cap \tau_5 = \varnothing$;
4. $\{\tau_1, \tau_2, \tau_3, \tau_4\}$: $\tau_1 \cap (\tau_2 \cup \tau_3 \cup \tau_4) = \varnothing$; $\tau_2 \cap \tau_3 \cap \tau_4 = \varnothing$;
5. $\{\tau_1, \tau_2, \tau_3\}$: $\tau_1 \cap \tau_2 = \tau_1 \cap \tau_3 = \tau_2 \cap \tau_3 = \varnothing$.

Idea of the proof

Proposition

We have $s_{\mathbb{R}}(K) \geqslant r$ *if and only if there exists a mapping* $\xi\colon\ \mathbb{Z}_2^r \setminus \{0\} \to N(K)$ *such that* $\xi(\boldsymbol{a}_1) \cap \cdots \cap \xi(\boldsymbol{a}_{2l+1}) = \varnothing$ *for any minimal linear dependence* $\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_{2l+1} = 0$.

Corollary

$s_{\mathbb{R}}(K) \geqslant 3$ *if and only if there exists a mapping* $\xi\colon\ \mathbb{Z}_2^3 \setminus \{0\} \to N(K)$ *such that* $\xi(\boldsymbol{a}) \cap \xi(\boldsymbol{b}) \cap \xi(\boldsymbol{c}) = \varnothing$ *for any triple of pairwise distinct vectors* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *with* $\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = 0$.

Problem”: Polytopes

Proposition (E.,13)

1. $s(P) = 1$ *if and only if* $P = \Delta^n$, $n > 0$ ($\Longleftrightarrow m - n = 1$).
2. *For any* $k \geqslant 2$ *there exists* P *with* $m - n = k$ *and* $s(P) = 2$;
3. *If* $s(P) = 2$, *then* $2 \leqslant m - n \leqslant 2 + \left[\frac{n}{2}\right]$. *In this case either* $P = I \times \Delta^n$, *or any two facets of* P *intersect. Moreover any* $(m - n - 2)$ *facets of* P *intersect. We have:*
   - *if* $m - n = 2$, *then* $P = \Delta^i \times \Delta^j$;
   - *if* $m - n = 3$, *then* $|N(K_P)| \geqslant 9$;
   - *if* $n = 2$, *then* $P = I \times I$;
   - *if* $n = 3$, *then* $P = I \times \Delta^2$.

Problem": Cyclic Polytopes

Definition (Cyclic polytope)

For $m \geqslant n+2$ and $t_1 < \cdots < t_m$ define cyclic polytope

$$C^n(t_1, \ldots, t_m) = \mathrm{Conv}\{(t_i, t_i^2, \ldots, t_i^n), i = 1, \ldots, m\}$$

Combinatorial type $C^n(m)$ does not depend on $t_1, \ldots, t_m$.

Theorem (E.,14)

For polytopes polar to cyclic we have

$$s(C^n(m)^\Delta) = 2 \text{ if and only if } 0 \leqslant m - n - 2 < \frac{\left[\frac{n}{2}\right] + 1}{3}.$$

Idea of the proof

Lemma

$C^{2l}(m)$ has cyclic symmetry and

$$C^{2l+1}(m)^{\Delta} = C^{2l}(m-1)^{\Delta}(2, 1, \ldots, 1).$$

Lemma

For $P = C^{2l}(m)^{\Delta}$ we have

$$N(K_P) = \{l + 1 \textit{ points on the circle with no neighboring }\}.$$

- The mapping $\xi\colon \mathbb{Z}_2^3 \setminus \{0\} \to N(K_P)$ associates the vector $\boldsymbol{a}$ to each vertex in $\xi(\boldsymbol{a})$.
- Each vertex in $[m]$ corresponds to no more than 3 vectors, so $7(l+1) \leqslant 3m$.
- Combining along the circle blocks

100	110	100	110	111	100	111
010	101	010	101	010	001	011
001	111	011	001	011	110	101

to build ξ we obtain that this condition is necessary and sufficient for $s(C^{2l}(m)^{\Delta}) \geqslant 3$.

Questions

- To classify simplicial complexes with $s(K) = 2$;
- To find "polytopal" criterion for $s(P) = 2$;
- To calculate $s(C^n(m)^\Delta)$ and $s_{\mathbb{R}}(C^n(m)^\Delta)$ for all m, n.

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Thank You for Your Attention!