On a class of quotient spaces of moment-angle complexes

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Content

Moment-Angle Complexes

Given an abstract simplicial complex $\mathcal{K}$ with vertex set $V(\mathcal{K}) = [m] = \{v_1, \cdots, v_m\}$ and a pair of spaces (X, A) with $A \subset X$, we can construct of a topological space $(X, A)^{\mathcal{K}}$ by:

$$(X, A)^{\mathcal{K}} = \bigcup_{\sigma \in \mathcal{K}} (X, A)^\sigma, \text{ where } (X, A)^\sigma = \prod_{v_i \in \sigma} X \times \prod_{v_i \notin \sigma} A.$$

$(X, A)^{\mathcal{K}}$ is called the *polyhedral product* or the *generalized moment-angle complex* of $\mathcal{K}$ and (X, A).

- $\mathcal{Z}_{\mathcal{K}} = (D^2, S^1)^{\mathcal{K}}$ is called the *moment-angle complex of* $\mathcal{K}$.
- $\mathbb{R}\mathcal{Z}_{\mathcal{K}} = (D^1, S^0)^{\mathcal{K}}$ is called the *real moment-angle complex of* $\mathcal{K}$.

Originally, $\mathcal{Z}_\mathcal{K}$ and $\mathbb{R}\mathcal{Z}_\mathcal{K}$ are introduced by Davis and Januszkiewicz (1991) in a different way.

- Let $P_\mathcal{K}$ denote the cone on the barycentric subdivision $\mathcal{K}'$ of $\mathcal{K}$.
- Any $(k-1)$-simplex $\sigma \in \mathcal{K}$ determines a subcomplex $F_\sigma \subset P_\mathcal{K}$. The polyhedron $P_\mathcal{K}$ together with its decomposition into "faces" $\{F_\sigma\}_{\sigma\in\mathcal{K}}$ is called a *simple polyhedral complex*.
- A map $\lambda : V(\mathcal{K}) \to (\mathbb{Z}_2)^r$ is called a *$(\mathbb{Z}_2)^r$-coloring of* $\mathcal{K}$.

  A map $\Lambda : V(\mathcal{K}) \to \mathbb{Z}^r$ is called a *$\mathbb{Z}^r$-coloring of* $\mathcal{K}$.

Basic Constructions

For any $\sigma \in \mathcal{K}$, let $V(\sigma)$ denote the vertex set of σ.

- $G_\lambda(\sigma) =$ the subgroup of $(\mathbb{Z}_2)^r$ generated by the set $\{\lambda(v) \,|\, v \in V(\sigma)\}$.
- $\widehat{G}_\Lambda(\sigma) =$ the toral subgroup of T^r corresponding to the subgroup of $\mathbb{Z}^r$ generated by $\{\Lambda(v) \,|\, v \in V(\sigma)\}$.

Define

$$X(\mathcal{K}, \lambda) := P_{\mathcal{K}} \times (\mathbb{Z}_2)^r / \sim \tag{1.1}$$

$$X(\mathcal{K}, \Lambda) := P_{\mathcal{K}} \times T^r / \sim \tag{1.2}$$

where $(p, g) \sim (p', g')$ whenever $p' = p \in F_\sigma$ and $g' g^{-1} \in G_\lambda(\sigma)$ or $\widehat{G}_\Lambda(\sigma)$ for some $\sigma \in \mathcal{K}$, respectively.

- If $\{\lambda(v_i)\,;\,1\le i\le m\}$ is a basis of $(\mathbb{Z}_2)^m$, $X(\mathcal{K},\lambda)\cong\mathbb{R}\mathcal{Z}_{\mathcal{K}}$.
- If $\{\Lambda(v_i)\,;\,1\le i\le m\}$ is a basis of $\mathbb{Z}^m$, $X(\mathcal{K},\Lambda)\cong\mathcal{Z}_{\mathcal{K}}$.

Denote the corresponding quotient maps by:

$$\pi_{\mathcal{K}}:P_{\mathcal{K}}\times(\mathbb{Z}_2)^m\to\mathbb{R}\mathcal{Z}_{\mathcal{K}},\quad \widehat{\pi}_{\mathcal{K}}:P_{\mathcal{K}}\times T^m\to\mathcal{Z}_{\mathcal{K}}$$

The <u>*canonical actions*</u> of $(\mathbb{Z}_2)^m$ on $\mathbb{R}\mathcal{Z}_{\mathcal{K}}$ and T^m on $\mathcal{Z}_{\mathcal{K}}$ are:

$$g'\cdot\pi_{\mathcal{K}}(p,g)=\pi_{\mathcal{K}}(p,g+g'),\;\; p\in P_{\mathcal{K}},\ g,g'\in(\mathbb{Z}_2)^m.$$

$$g'\cdot\widehat{\pi}_{\mathcal{K}}(p,g)=\widehat{\pi}_{\mathcal{K}}(p,gg'),\;\; p\in P_{\mathcal{K}},\ g,g'\in T^m.$$

Some Known Results on Moment-Angle Complexes

Let $\mathbf{k}$ denote $\mathbb{Z}$ or a field.

Theorem (Buchstaber-Panov)

The cohomology ring $H^*(\mathcal{Z}_\mathcal{K}, \mathbf{k})$ of $\mathcal{Z}_\mathcal{K}$ is isomorphic to the Tor algebra $\mathrm{Tor}_{\mathbf{k}[v_1,\cdots,v_m]}(\mathbf{k}(\mathcal{K}), \mathbf{k})$, where $\mathbf{k}(\mathcal{K})$ is the *face ring* of $\mathcal{K}$.

$$\mathbf{k}(\mathcal{K}) = \mathbf{k}[v_1, \cdots, v_m]/\mathcal{I}_\mathcal{K}$$

where $\mathcal{I}_\mathcal{K}$ is the ideal generated by the monomials $v_{i_1} \cdots v_{i_s}$ for which $\{v_{i_1}, \cdots, v_{i_s}\}$ does not span a simplex of $\mathcal{K}$.

Remark: There is no analogous isomorphism for the cohomology ring $H^*(\mathbb{R}\mathcal{Z}_\mathcal{K}, \mathbf{k})$ of $\mathbb{R}\mathcal{Z}_\mathcal{K}$.

Hochster's formula

For any coefficients $\mathbf{k}$, we have:

$$H^q(\mathbb{R}\mathcal{Z}_{\mathcal{K}};\mathbf{k}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^{q-1}(\mathcal{K}_J;\mathbf{k}),\ q\geq 0,$$

$$H^q(\mathcal{Z}_{\mathcal{K}};\mathbf{k}) \cong \bigoplus_{J\subset[m]} \widetilde{H}^{q-|J|-1}(\mathcal{K}_J;\mathbf{k}),\ q\geq 0,$$

where $\mathcal{K}_J$ is the full subcomplex of $\mathcal{K}$ obtained by restricting to J.

Stable decompositions (Bahri-Bendersky-Cohen-Gitler)

There are homotopy equivalences:

$$\mathbf{\Sigma}(\mathbb{R}\mathcal{Z}_{\mathcal{K}}) \simeq \bigvee_{J\subset[m]} \mathbf{\Sigma}^2(\mathcal{K}_J),\quad \mathbf{\Sigma}(\mathcal{Z}_{\mathcal{K}}) \simeq \bigvee_{J\subset[m]} \mathbf{\Sigma}^{|J|+2}(\mathcal{K}_J).$$

Quotient Spaces of (Real) Moment-Angle Complexes

For a subgroup $H \subset (\mathbb{Z}_2)^m$ or $\widehat{H} \subset T^m$, we get quotient spaces

$$\mathbb{R}\mathcal{Z}_{\mathcal{K}}/H, \quad \mathcal{Z}_{\mathcal{K}}/\widehat{H}$$

where H and $\widehat{H}$ act on $\mathbb{R}\mathcal{Z}_{\mathcal{K}}$ and $\mathcal{Z}_{\mathcal{K}}$ through the canonical actions, respectively.

The study of such quotients space are mainly done when $\mathcal{K}$ is a simplicial sphere that is dual to a simple convex polytope.

Let $\mathcal{K}^P$ be a simplicial sphere dual to an n-dimensional simple convex polytope P with m facets.

– $\mathcal{Z}_P = \mathcal{Z}_{\mathcal{K}^P}$ is called the *moment-angle manifold* of P.

– $\mathbb{R}\mathcal{Z}_P = \mathbb{R}\mathcal{Z}_{\mathcal{K}^P}$ is called the *real moment-angle manifold* of P.

Some Examples

- If $H \cong (\mathbb{Z}_2)^{m-n}$ is a subgroup of $(\mathbb{Z}_2)^m$ that acts freely on $\mathbb{R}\mathcal{Z}_P$ through the canonical action, $\mathbb{R}\mathcal{Z}_P/H$ is called a *small cover* over P.
- If $\widehat{H} \cong T^{m-n}$ is a subtorus of T^m that acts freely on $\mathcal{Z}_P$ through the canonical action, $\mathcal{Z}_P/\widehat{H}$ is called a *quasitoric manifold* over P.

Small covers and quasitoric manifolds are introduced by Davis and and Januszkiewicz (1991). Many topological properties (e.g. cohomology groups, characteristic classes) of these spaces are known.

- For a general subtorus $\widehat{H} \subset T^m$ which acts freely on $\mathcal{Z}_P$, there is an algebra isomorphism (by Buchstaber-Panov) between the cohomology ring $H^*(\mathcal{Z}_P/\widehat{H};\mathbf{k})$ and

$$\mathrm{Tor}_{\mathbf{k}[u_1,\cdots,u_{m-r}]}(\mathbf{k}[\mathcal{K}^P];\mathbf{k})$$

where $\dim(\widehat{H}) = r$ and the $\mathbf{k}[\mathcal{K}^P]$ is equipped with a $\mathbf{k}[u_1,\cdots,u_{m-r}]$-module structure determined by $\widehat{H}$.

Spaces associated to a partition of $V(\mathcal{K})$

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a *partition* of $V(\mathcal{K})$, i.e. α_i's are disjoint subsets of $V(\mathcal{K})$ with $\alpha_1 \cup \cdots \cup \alpha_k = V(\mathcal{K})$.

- Let λ_α be a $(\mathbb{Z}_2)^k$-coloring of $\mathcal{K}$ defined by $\lambda_\alpha(\alpha_i) = e_i$, $1 \le i \le k$, where $\{e_1, \cdots, e_k\}$ is a basis of $(\mathbb{Z}_2)^k$.
- Let Λ_α be a $\mathbb{Z}^k$-coloring of $\mathcal{K}$ defined by $\Lambda_\alpha(\alpha_i) = \hat{e}_i$, $1 \le i \le k$, where $\{\hat{e}_1, \cdots, \hat{e}_k\}$ is a basis of $\mathbb{Z}^k$.

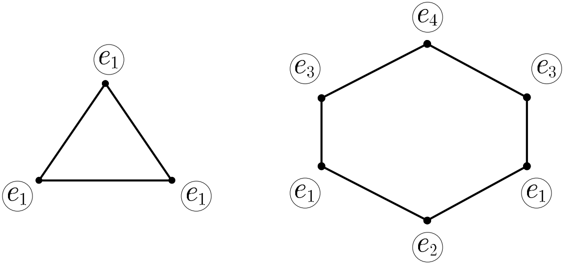
Then we obtain two spaces $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$ which are determined by $\mathcal{K}$ and α.

Example

Let α^* denote the *trivial partition* of $V(\mathcal{K})$, i.e. $\alpha^* = (\alpha_1, \cdots, \alpha_m)$ where each $\alpha_j = \{v_j\}$ consists of only one vertex of $\mathcal{K}$. Then

$$X(\mathcal{K},\lambda_{\alpha^*}) = \mathbb{R}\mathcal{Z}_{\mathcal{K}}, \quad X(\mathcal{K},\Lambda_{\alpha^*}) = \mathcal{Z}_{\mathcal{K}}.$$

Examples of $X(\mathcal{K}, \lambda_\alpha)$



- The left picture gives a 2-dimensional sphere.
- The right picture gives a closed connected orientable surface of genus 5.

Cohomology Groups

For any simplex $\sigma \in \mathcal{K}$, let $V(\sigma) \subset [m]$ denote the vertex set of σ.

- Define $\mathrm{rank}(\sigma) := |V(\sigma)| = \dim(\sigma) + 1$.
- Define $\mathrm{I}_\alpha(\sigma) := \{i \in [k]\,;\, V(\sigma) \cap \alpha_i \neq \varnothing\} \subset [k]$.

For any subset $\mathrm{L} \subset [k] = \{1, \cdots, k\}$, define

$\mathcal{K}_{\alpha,\mathrm{L}} :=$ the subcomplex of $\mathcal{K}$ consisting of $\{\sigma \in \mathcal{K}\,;\, \mathrm{I}_\alpha(\sigma) \subset \mathrm{L}\}$.

Theorem 1

- $H^q(X(\mathcal{K},\lambda_\alpha);\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[k]} \widetilde{H}^{q-1}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k})$ for any $q \geq 0$.
- $H^q(X(\mathcal{K},\Lambda_\alpha);\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[k]} \widetilde{H}^{q-|\mathrm{L}|-1}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k})$ for any $q \geq 0$.

Stable Decompositions

Theorem 2

There are homotopy equivalences:

$$\mathbf{\Sigma}(X(\mathcal{K},\lambda_\alpha)) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^2(\mathcal{K}_{\alpha,\mathrm{L}}),$$

$$\mathbf{\Sigma}(X(\mathcal{K},\Lambda_\alpha)) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^{|\mathrm{L}|+2}(\mathcal{K}_{\alpha,\mathrm{L}}).$$

Remark: Theorem 1 also follows from the stable decompositions of $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$ in Theorem 2. But we can not obtain the cohomology ring structures of $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$ from Theorem 2.

Non-degenerate Partitions

A partition α of $V(\mathcal{K})$ is called *non-degenerate* if

$$|\mathrm{I}_\alpha(\sigma)| = \mathrm{rank}(\sigma) \text{ for any simplex } \sigma \in \mathcal{K}.$$

In particular, the trivial partition α^* of $V(\mathcal{K})$ is always non-degenerate.

It is easy to see that the following statements are equivalent.

- $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ is a non-degenerate partition of $V(\mathcal{K})$.
- the two vertices of any 1-simplex of $\mathcal{K}$ belong to different α_i.
- The coloring λ_α or Λ_α on $\mathcal{K}$ is non-degenerate.

Theorem 3

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be a non-degenerate partition of the vertex set of $\mathcal{K}$. Then with respect to some properly defined multi-gradings,

- there is an isomorphism of multigraded $\mathbb{Z}_2$-modules from $H^*(X(\mathcal{K},\lambda_\alpha);\mathbb{Z}_2)$ to $\mathrm{Tor}_{\mathbb{Z}_2[u_1,\cdots,u_k]}(\mathbb{Z}_2[\mathcal{K}];\mathbb{Z}_2)$.
- there is an isomorphism of multigraded $\mathbb{Z}_2$-algebras from $H^*(X(\mathcal{K},\Lambda_\alpha);\mathbb{Z}_2)$ to $\mathrm{Tor}_{\mathbb{Z}_2[u_1,\cdots,u_k]}(\mathbb{Z}_2[\mathcal{K}];\mathbb{Z}_2)$.

Here the $\mathbb{Z}_2[u_1,\cdots,u_k]$-module structure on the face ring $\mathbb{Z}_2[\mathcal{K}] = \mathbb{Z}_2[v_1,\cdots,v_m]/\mathcal{I}_\mathcal{K}$ is given by

$$\mathbb{Z}_2[u_1,\cdots,u_k] \longrightarrow \mathbb{Z}_2[v_1,\cdots,v_m], \quad u_i \mapsto \sum_{v_j\in\alpha_i} v_j .$$

Face Ring - the second definition

The face ring $\mathbf{k}[\mathcal{K}]$ can be equivalently defined as the quotient algebra

$$\mathbf{k}[\mathcal{K}] := \mathbf{k}[\,\mathbf{v}_\sigma : \sigma \in \mathcal{K}\,]/\mathcal{I}'_\mathcal{K}$$

where $\mathcal{I}'_\mathcal{K}$ is the ideal generated by all the elements of the form

$$\mathbf{v}_{\hat{\mathbf{0}}} - 1, \quad \mathbf{v}_\sigma \mathbf{v}_\tau - \mathbf{v}_{\sigma\wedge\tau} \cdot \mathbf{v}_{\sigma\vee\tau}.$$

– $\sigma \wedge \tau$ denotes the maximal common face of σ and τ.

– $\sigma \vee \tau$ denotes the minimal face of $\mathcal{K}$ that contains both σ and τ.

– If there is no simplex contains both σ and τ, we let $\mathbf{v}_{\sigma\vee\tau} = 0$.

A Differential Graded Algebra

For a general partition α of $V(\mathcal{K})$ (may not be non-degenerate), we can describe $H^*(X(\mathcal{K},\lambda_\alpha);\mathbb{Z}_2)$ and $H^*(X(\mathcal{K},\Lambda_\alpha);\mathbb{Z}_2)$ in terms of some differential graded algebra determined by $\mathcal{K}$ and α defined as follows.

Define a differential d_α on $\Lambda_{\mathbb{Z}_2}[t_1,\cdots,t_k]\otimes\mathbb{Z}_2[\mathcal{K}]$ by:

$$d_\alpha(t_i) := \sum_{v_j\in\alpha_i}\mathbf{v}_j,\ \ 1\leq i\leq k,$$

$$d_\alpha(\mathbf{v}_\sigma) := \sum_{\substack{\sigma\subset\omega,\,\mathrm{I}_\alpha(\omega)=\mathrm{I}_\alpha(\sigma)\\ \dim(\omega)=\dim(\sigma)+1}}\mathbf{v}_\omega.$$

Note: d_α is zero on $\mathbf{k}[\mathcal{K}]$ if and only if α is non-degenerate.

Theorem 4

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ be an arbitrary partition of $V(\mathcal{K})$. Then with respect to some properly defined multi-gradings,

- there is an isomorphism of multigraded $\mathbb{Z}_2$-modules from $H^*(X(\mathcal{K},\lambda_\alpha);\mathbb{Z}_2)$ to $H^*(\Lambda_{\mathbb{Z}_2}[t_1,\cdots,t_k]\otimes\mathbb{Z}_2[\mathcal{K}], d_\alpha)$.
- there is an isomorphism of multigraded $\mathbb{Z}_2$-algebras from $H^*(X(\mathcal{K},\Lambda_\alpha);\mathbb{Z}_2)$ to $H^*(\Lambda_{\mathbb{Z}_2}[t_1,\cdots,t_k]\otimes\mathbb{Z}_2[\mathcal{K}], d_\alpha)$.

Remark: When α is a non-degenerate partition of $V(\mathcal{K})$,

$$H^*(\Lambda_{\mathbb{Z}_2}[t_1,\cdots,t_k]\otimes\mathbb{Z}_2[\mathcal{K}], d_\alpha) \cong \mathrm{Tor}_{\mathbb{Z}_2[u_1,\cdots,u_k]}(\mathbb{Z}_2[\mathcal{K}];\mathbb{Z}_2).$$

So Theorem 3 is a special case of Theorem 4.

Pullback From the Linear Model

Balanced Simplicial Complex

An $(n-1)$-dimensional simplicial complex $\mathcal{K}$ is called *balanced* if there exists a map $\phi : V(\mathcal{K}) \to [n] = \{1, \cdots, n\}$ such that if $\{v, v'\}$ is an edge of $\mathcal{K}$, then $\phi(x) \neq \phi(y)$. We call ϕ an *n-coloring* on $\mathcal{K}$.

Let $\{e_1, \cdots, e_n\}$ be a basis of $(\mathbb{Z}_2)^n$. Then ϕ uniquely determines a non-degenerate $(\mathbb{Z}_2)^n$-coloring

$$\lambda^\phi : V(\mathcal{K}) \to (\mathbb{Z}_2)^n, \quad \lambda^\phi(v) = e_{\phi(v)}.$$

The space $X(\mathcal{K}, \lambda^\phi)$ is called a *pullback from the linear model* by Davis and Januszkiewicz.

Note that $\alpha_\phi := \{\phi^{-1}(1), \cdots \phi^{-1}(n)\}$ is a partition of $V(\mathcal{K})$. So

$$X(\mathcal{K},\lambda^\phi) = X(\mathcal{K},\lambda_{\alpha_\phi}).$$

- by Theorem 1, the cohomology groups of $X(\mathcal{K},\lambda^\phi)$ are:

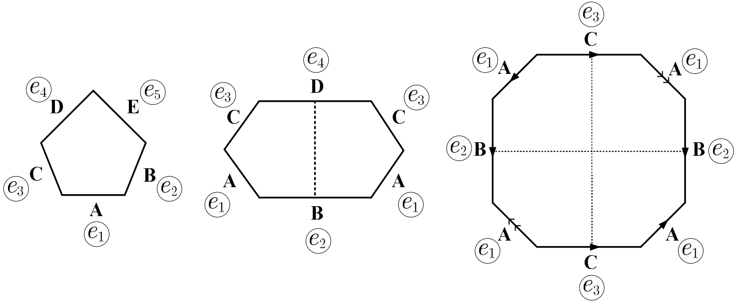
$$H^q(X(\mathcal{K},\lambda^\phi);\mathbf{k}) = \sum_{\mathrm{L}\subset[n]} \widetilde{H}^{q-1}(\mathcal{K}_{\alpha_\phi,\mathrm{L}};\mathbf{k}),\ \forall q \geq 0.$$

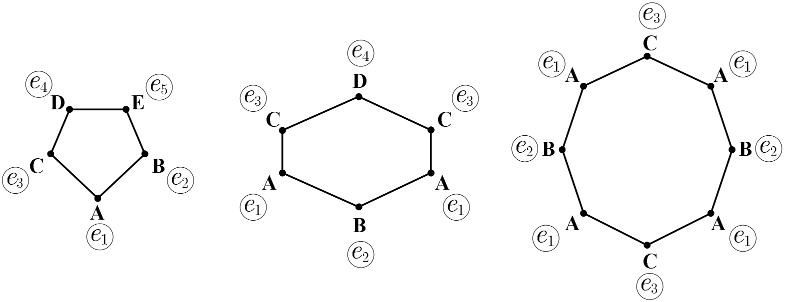
- By Theorem 2, we have a homotopy equivalence

$$\mathbf{\Sigma}(X(\mathcal{K},\lambda^\phi)) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^2(\mathcal{K}_{\alpha_\phi,\mathrm{L}}).$$

Moment-Angle Complexes as Partial Quotients

The (real) moment-angle complex of $\mathcal{K}$ can be viewed as a partial quotient of the (real) moment-angle complex some larger simplicial complexes.

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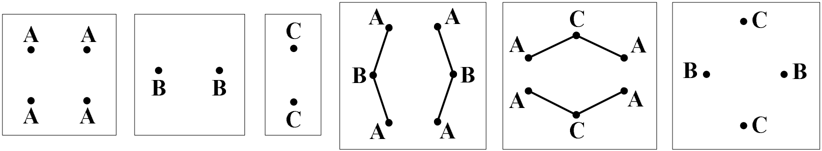
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Moment-Angle Complexes of Simplicial Posets

Simplicial Poset

A poset (partially ordered set) $\mathcal{S}$ with the order relation $\leq$ is called *simplicial* if it has an initial element $\hat{0}$ and for each $\sigma \in \mathcal{S}$ the lower segment

$$[\hat{0}, \sigma] = \{\tau \in \mathcal{S} : \hat{0} \leq \tau \leq \sigma\}$$

is the face poset of a simplex. We refer to $\sigma \in \mathcal{S}$ as a simplex of dimension n if $[\hat{0}, \sigma]$ is the face poset of an n-simplex.

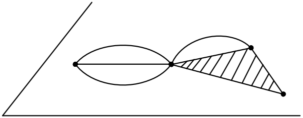
for each $\sigma \in \mathcal{S}$ we assign a geometric simplex Δ^σ whose face poset is $[\hat{0}, \sigma]$, and glue these geometric simplices together according to the order relation in $\mathcal{S}$. We obtain a cell complex $|\mathcal{S}|$ called the *geometric realization* of $\mathcal{S}$.

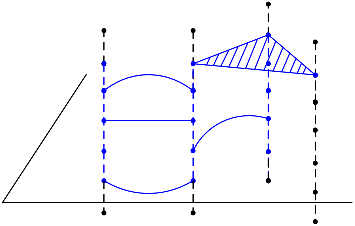
For a simplicial poset $\mathcal{S}$, the moment-angle complex is defined as

$$\mathcal{Z}_{\mathcal{S}} := \operatorname*{colim}_{\sigma\in\mathcal{S}} (D^2, S^1)^\sigma \quad \text{(by Lü-Panov)}$$

Proposition 1:

For any finite simplicial poset $\mathcal{S}$, there always exists a finite simplicial complex $\mathcal{K}$ and a partition α of $V(\mathcal{K})$ so that $\mathcal{Z}_{\mathcal{S}}$ is homotopy equivalent to $X(\mathcal{K},\Lambda_\alpha)$.





Theorem (Lü and Panov)

There is a multi-graded algebra isomorphism:

$$H^*(\mathcal{Z}_\mathcal{S};\mathbb{Z}) \cong \mathrm{Tor}_{\mathbb{Z}[v_1,\cdots,v_m]}(\mathbb{Z}[\mathcal{S}],\mathbb{Z})$$

where $m =$ the number of 0-simplices of $\mathcal{S}$.

Then Proposition 1 implies that there is a multi-graded algebra isomorphism from $\mathrm{Tor}_{\mathbb{Z}_2[v_1,\cdots,v_m]}(\mathbb{Z}_2[\mathcal{S}],\mathbb{Z}_2)$ to

$$H^*(X(\mathcal{K},\Lambda_\alpha);\mathbb{Z}_2) \cong H^*(\Lambda_{\mathbb{Z}_2}[t_1,\cdots,t_m]\otimes\mathbb{Z}_2[\mathcal{K}],d_\alpha)$$

for some simplicial complex $\mathcal{K}$ and a partition $\alpha = \{\alpha_1,\cdots,\alpha_m\}$ of $V(\mathcal{K})$.

Some Notations

- For any simplex $\sigma \in \mathcal{K}$, let C_σ denote the cone of the barycentric subdivision of σ. Then C_σ is a cube with
$$\dim(C_\sigma) = \operatorname{rank}(\sigma).$$
- $P_\mathcal{K}$ has a natural cubical cell decomposition
$$P_\mathcal{K} = \bigcup_{\sigma \in \mathcal{K}} C_\sigma$$
- Let $\Delta^{[m]}$ be the m-simplex with vertex set $[m]$. For a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $[m]$, let
$$\Delta^{\alpha_i} = \text{ the face of } \Delta^{[m]} \text{ whose vertex set is } \alpha_i.$$

Pullback Construction

The cubical complex $P_{\mathcal{K}}$ naturally embeds into the m-cube $P_{\Delta^{[m]}}$, which gives the following commutative diagrams

$$\begin{array}{ccc} X(\mathcal{K},\lambda_\alpha) & \longrightarrow & X(\Delta^{[m]},\lambda_\alpha) \\ \downarrow & & \downarrow \\ P_{\mathcal{K}} & \longrightarrow & P_{\Delta^{[m]}} \end{array} \qquad \begin{array}{ccc} X(\mathcal{K},\Lambda_\alpha) & \longrightarrow & X(\Delta^{[m]},\Lambda_\alpha) \\ \downarrow & & \downarrow \\ P_{\mathcal{K}} & \longrightarrow & P_{\Delta^{[m]}} \end{array}$$

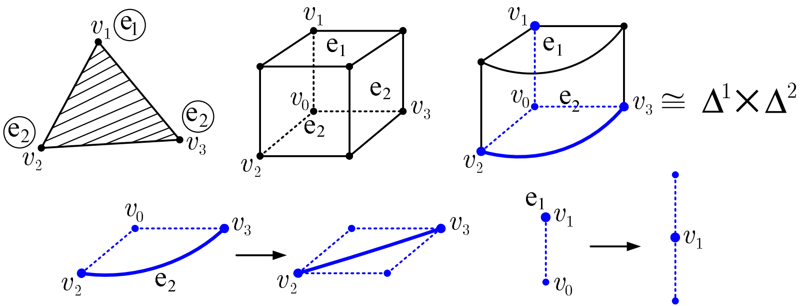
where the vertical ones are orbit maps for the $(\mathbb{Z}_2)^k$ and T^k actions, respectively.

$$X(\Delta^{[m]},\lambda_\alpha) \cong \prod_{i\in[k]} S^0 * \Delta^{\alpha_i}, \quad X(\Delta^{[m]},\Lambda_\alpha) \cong \prod_{i\in[k]} S^1 * \Delta^{\alpha_i}.$$

Example

Let $m=3$, $\alpha=(\{1\},\{2,3\})$. Then $k=2$ and

$$X(\Delta^{[m]},\lambda_\alpha)\cong\prod_{i\in[k]}S^0*\Delta^{\alpha_i}\cong(S^0*\Delta^1)\times(S^0*\Delta^2)$$



Decompositions of $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$

- For any simplex $\sigma \in \mathcal{K}$, define

$$\mathbf{D}_\alpha(\sigma) = \prod_{i\in \mathrm{I}_\alpha(\sigma)} S^0_{(i)} * (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i\in[k]\setminus \mathrm{I}_\alpha(\sigma)} S^0_{(i)},$$

$$\widehat{\mathbf{D}}_\alpha(\sigma) = \prod_{i\in \mathrm{I}_\alpha(\sigma)} S^1_{(i)} * (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i\in[k]\setminus \mathrm{I}_\alpha(\sigma)} S^1_{(i)}$$

where $S^0_{(i)}$ and $S^1_{(i)}$ are the copy of S^0 and S^1 associated to $i \in [k]$. We can show that:

$$X(\mathcal{K},\lambda_\alpha) = \bigcup_{\sigma\in\mathcal{K}} \mathbf{D}_\alpha(\sigma), \quad X(\mathcal{K},\Lambda_\alpha) = \bigcup_{\sigma\in\mathcal{K}} \widehat{\mathbf{D}}_\alpha(\sigma)$$

Then using the minimal cell decomposition of S^0 and S^1, we obtain cell decompositions of $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$.

Some Generalizations

- For a partition $\alpha=\{\alpha_1,\cdots,\alpha_k\}$ of $V(\mathcal{K})$ and a sequence of spheres $\mathbb{S}=(S^{d_1},\cdots,S^{d_k})$, define

$$X(\mathcal{K},\alpha,\mathbb{S})=\bigcup_{\sigma\in\mathcal{K}}\Big(\prod_{i\in\mathrm{I}_\alpha(\sigma)}S^{d_i}*(\sigma\cap\Delta^{\alpha_i})\times\prod_{i\in[k]\setminus\mathrm{I}_\alpha(\sigma)}S^{d_i}\Big)$$

$$\subset\prod_{i\in[k]}S^{d_i}*\Delta^{\alpha_i}$$

Remark: When α^* is the trivial partition of $V(\mathcal{K})$, $X(\mathcal{K},\alpha^*,\mathbb{S})$ is nothing but the polyhedral-product $\mathcal{K}^{(\mathbb{D},\mathbb{S})}$ where

$$(\mathbb{D},\mathbb{S})=\{(D^{d_1+1},S^{d_1}),\cdots,(D^{d_k+1},S^{d_k})\}.$$

Parallel Theorems for $X(\mathcal{K},\alpha,\mathbb{S})$

Theorem 1*

For any coefficients $\mathbf{k}$, there is a $\mathbf{k}$-module isomorphism:

$$H^q(X(\mathcal{K},\alpha,\mathbb{S});\mathbf{k}) \cong \bigoplus_{\mathrm{L}\subset[k]} \widetilde{H}^{q-1-\sum_{i\in\mathrm{L}} d_i}(\mathcal{K}_{\alpha,\mathrm{L}};\mathbf{k}),\ \ \forall q\geq 0.$$

Theorem 2*

There is a homotopy equivalence

$$\mathbf{\Sigma}(X(\mathcal{K},\alpha,\mathbb{S})) \simeq \bigvee_{\mathrm{L}\subset[k]} \mathbf{\Sigma}^{(\sum_{i\in\mathrm{L}} d_i)+2}(\mathcal{K}_{\alpha,\mathrm{L}}).$$

Theorem 4*

Let α be an arbitrary partition of $V(\mathcal{K})$.

(i) For any family $\mathbb{S} = \{S^{d_1}, \cdots, S^{d_k}\}$, $H^*(X(\mathcal{K}, \alpha, \mathbb{S}); \mathbb{Z}_2)$ is isomorphic to $H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}], d_\alpha)$ as multigraded $\mathbb{Z}_2$-modules.

(ii) For $\mathbb{S} = (S^{d_1}, \cdots, S^{d_k})$ with $d_i \geq 1$, $i = 1, \cdots, k$, $H^*(X(\mathcal{K}, \alpha, \mathbb{S}); \mathbb{Z}_2)$ is isomorphic to $H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}], d_\alpha)$ as multigraded $\mathbb{Z}_2$-algebras.

In particular when α is non-degenerate,

$$H^*(\Lambda_{\mathbb{Z}_2}[t_1, \cdots, t_k] \otimes \mathbb{Z}_2[\mathcal{K}], d_\alpha) \cong \mathrm{Tor}_{\mathbb{Z}_2[u_1, \cdots, u_k]}(\mathbb{Z}_2[\mathcal{K}]; \mathbb{Z}_2).$$

- For a partition $\alpha = \{\alpha_1, \cdots, \alpha_k\}$ of $V(\mathcal{K})$ and a sequence of spaces $\mathbb{A} = (A_1, \cdots, A_k)$, define

$$X(\mathcal{K},\alpha,\mathbb{A}) = \bigcup_{\sigma\in\mathcal{K}} \Big(\prod_{i\in \mathrm{I}_\alpha(\sigma)} A_i * (\sigma \cap \Delta^{\alpha_i}) \times \prod_{i\in[k]\backslash \mathrm{I}_\alpha(\sigma)} A_i \Big)$$

$$\subset \prod_{i\in[k]} A_i * \Delta^{\alpha_i}.$$

In particular, for the trivial partition α^* of $V(\mathcal{K}) = [m]$, $X(\mathcal{K},\alpha^*,\mathbb{A})$ = the polyhedral product $(\mathrm{Cone}(\mathbb{A}),\mathbb{A})^{\mathcal{K}}$ where

$$(\mathrm{Cone}(\mathbb{A}),\mathbb{A}) = \{(\mathrm{Cone}(A_1),A_1),\cdots,(\mathrm{Cone}(A_m),A_m)\}.$$

Theorem 5

There is a homotopy equivalence

$$\boldsymbol{\Sigma}(X(\mathcal{K}, \alpha, \mathbb{A})) \simeq \boldsymbol{\Sigma}\Big(\bigvee_{\mathrm{L} \subset [k]} |\mathcal{K}_{\alpha,\mathrm{L}}| * \bigwedge_{i \in \mathrm{L}} A_i \Big).$$

Then we can compute the cohomology groups of $X(\mathcal{K}, \alpha, \mathbb{A})$ from this decomposition.

Comparison

- The building blocks of $X(\mathcal{K},\lambda_\alpha)$ and $X(\mathcal{K},\Lambda_\alpha)$ are spaces obtained by mixtures of Cartesian products and joins of some simple spaces (simplices, S^0 and S^1).
- The building blocks of polyhedral products $(X,A)^{\mathcal{K}}$ only involve Cartesian products of spaces.

In addition, we have the following notions.

- *Polyhedral join* (introduced by Ayzenberg)
- *Polyhedral smash product* (introduced by Bahri, Bendersky, Cohen and Gitler)

Explore the spaces with building blocks involving mixtures of Cartesian products, joins and smash products ....

$$\cdots \times \cdots \bigwedge \cdots * \cdots \bigwedge \cdots$$

End of Talk

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