Integrable systems, toric degenerations and Newton-Okounkov bodies

Kiumars Kaveh
(joint with Megumi Harada)

Torus actions and applications
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Symplectic geometry

Symplectic geometry = modern formulation of classical mechanics

- M manifold, $\dim_{\mathbb{R}}(M) = 2n$
- ω non-degenerate 2-form
- $f \in C^\infty(M) \mapsto \xi_f \in Vec(M)$
- $\omega(\xi_f, \cdot) = df(\cdot)$
- M phase space of the system
- f = Hamiltonian of the system (energy)
- Flow of ξ_f = evolution of the system

Main example

- $M = \mathbb{R}^{2n}$, coordiantes q_i, p_i position and momentum
- $\omega = \sum_i dp_i \wedge dq_i$
- $f = \frac{1}{2}\sum_i (q_i^2 + p_i^2) =$ kinetic + potential energy
- Flow of ξ_f describes n- *uncoupled harmonic oscillators*

Another famous example

- M = sphere S^2
- ω = surface area
- $f : M \to \mathbb{R}$ height function

Observation (Archemedes)

Hamiltonian vector field of f generates S^1 action on M (rotation of sphere)

Related to theorem of Archemedes (sphere inscribed in cylinder on his tomb stone)

Conservation laws

$f, g \in C^\infty(M)$

Definition

$\{f, g\} := \omega(\xi_f, \xi_g)$ *Poisson bracket*

$[\xi_f, \xi_g] = -\xi_{\{f,g\}}$

Theorem

If $\{f, g\} = 0$ *then* g *constant along flow of* ξ_f.

Implies conservation laws in classical mechanics

Completely integrable system

(M, ω) symplectic manifold, $\dim_{\mathbb{R}}(M) = 2n$

Definition

$f_1, \ldots, f_n \in C^\infty(M)$ completely integrable system:

- $\{f_i, f_j\} = 0, \ \forall i, j$
- $df_1, \ldots, df_n$ are linearly independent a.e. on M.

Completely integrable systems are very well-behaved dynamical systems

Toric varieties

- $T = (S^1)^n$, $T_{\mathbb{C}} = (\mathbb{C}^*)^n$
- Fix $\mathcal{A} = \{\alpha_0, \ldots, \alpha_N\} \subset \mathbb{Z}^n$
- $T_{\mathbb{C}} \curvearrowright \mathbb{C}P^N$ by:
  $t \cdot (z_0 : \cdots : z_N) := (t^{\alpha_0} z_0 : \cdots : t^{\alpha_N} z_N)$, where
  $t^{\alpha} := t_1^{a_1} \cdots t_n^{a_n}$
- Assume stabilizer of $(1 : \cdots : 1)$ is trivial,
  $X_{\mathcal{A}} := \overline{T_{\mathbb{C}} \cdot (1 : \cdots : 1)}$ (projective toric variety possibly singular)
- $T_{\mathbb{C}} \hookrightarrow X_{\mathcal{A}}$, $t \mapsto t \cdot (1 : \cdots : 1)$

Moment map

- T preserves Fubini-Study symplectic form on $\mathbb{C}P^N$ (and hence X)
- $X_{\mathcal{A}}$ Hamiltonian T-space (when smooth) with moment map $\mu = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$:

$$\mu(t) = (const.)\frac{\sum_i \alpha_i |t|^{2\alpha_i}}{\sum_i |t|^{2\alpha_i}}$$

- $\mu(X) = \mathrm{conv}(\mathcal{A})$ (Newton polytope)
- Hamiltonian vector fields of the f_i generate the T action
- $f_1, \ldots, f_n$ completely integrable system

Baby example

- $\mathcal{A} = \{0, 2, 3\} \subset \mathbb{Z}$
- $\mathbb{C}^* \curvearrowright \mathbb{C}P^2$ by:

$$t \cdot (x : y : z) = (t^2 x : t^3 y : z)$$

- $X_{\mathcal{A}} = \{y^2 z = x^3\} \subset \mathbb{C}P^2$, cuspidal cubic
- $\mu(t) = (const.)\frac{2|t|^4+3|t|^6}{1+|t|^4+|t|^6}$, $\mu(X_{\mathcal{A}}) = [0, 3]$

Gelfand-Zetlin integrable system

- $X = \{V_1 \subset \cdots V_{n-1} \subset \mathbb{C}^n\}$ flag variety
- Fix $\lambda = (\lambda_1 < \cdots < \lambda_n)$. Then X can be identified with all Hermitian matrices with eignevalues λ
- i.e. X coadjoint orbit O_λ for $SU(n)$. Equip X with Kostant-Kirillov-Souriau symplectic form
- $X \hookrightarrow \mathbb{P}(V_\lambda)$

Gelfand-Zetlin integrable system

Let $x = n \times n$ Hermitian matrix representing a point in X. Let $\mu_{1,i} \leq \cdots \leq \mu_{n-i,i}$ eigenvalues of x_i, $(n-i) \times (n-i)$ submatrix of x. One has:

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{n-1} & & \lambda_n \\
& \mu_{1,1} & & \cdots & & \cdots & & \mu_{n-1,1} & \\
& & \mu_{1,2} & & \cdots & & \mu_{n-2,2} & & \\
& & & \cdots & & \cdots & & & \\
& & & & \mu_{1,n-1} & & & &
\end{array}
$$

Gelfand-Zetlin integrable system

Theorem (Guillemin-Sternberg)

$\mu = (\mu_{i,j})$ gives a completely integrable system on a dense open subset of X. The image of μ is GZ polytope $\Delta(\lambda)$

Completely integrable system

Let X algebraic variety, $\dim_{\mathbb{C}}(X) = n$, ω symplectic form on X^{smooth}.

Definition

$f_1, \ldots, f_n$ *completely integrable system* on X:

- $f_1, \ldots, f_n$ are continuous on all of X.
- $\exists$ open dense $U \subset X^{smooth}$ s.t. $f_1, \ldots, f_n$ differentiable on U, and $df_1, \ldots, df_n$ linearly independent on U.
- $f_1, \ldots, f_n$ pairwise Poisson-commute on U.

Newton-Okounkov bodies

$X \subset \mathbb{C}P^N$ projective variety, $\dim(X) = n$

- $A = \bigoplus_{k \geq 0} A_k$ homog. coordinate ring of X
- Fix a valuation v on $\mathbb{C}(X)$ and $0 \neq h \in A_1$
- $S(A) := \bigcup_{k>0} \{(v(f/h^k), k) \mid f \in A_k \setminus \{0\}\}.$
- $\Delta = \Delta(A, v, h) =$ Newton-Okounkov body

Generalized moment maps?

Question

Is there a "good" continuous map $\mu : X \to \Delta$*?*

Integrable systems on varieties

Suppose $S(A)$ finitely generated semigroup. Then $\Delta = \Delta(A)$ a rational polytope. Equip X with Kähler form from $\mathbb{C}P^N$

Theorem (Harada-K. 2012)

$\exists$ completely integrable system $f_1, \ldots, f_n$ on X. Moreover

- *Image of $\mu = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ coincides with Δ.*
- *$f_1, \ldots, f_n$ generate a torus action on U*
- *$\mu^{-1}(\Delta^\circ) \subset U$*

Contains many examples

Examples

- Flag varieties and Schubert varieties of all reductive groups (GZ polytopes, string polytopes)
- Spherical varieties (string polytopes of Okounkov and Brion-Alexeev)
- Weight varieties e.g. polygon spaces (slices of GZ polytopes and string polytopes)

Toric degeneration

X projective variety

Definition

$\pi : \mathfrak{X} \to \mathbb{C}$ *toric degeneration* of X if:

- $\pi : \mathfrak{X} \to \mathbb{C}$ is a flat family of irreducible varieties
- The family is trivial over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with fiber isomorphic to X
- The special fiber X_0 is a toric variety

Gradient-Hamiltonian vector field

$\mathfrak{X}$ = Kähler manifold, $\pi : \mathfrak{X} \to \mathbb{C}$ holomorphic, $X_t := \pi^{-1}(t)$

- $\nabla(Re(\pi)) = -\xi_{Im(\pi)}$ (Cauchy-Riemann)
- Define $V_\pi := -\frac{\nabla(Re(\pi))}{\|\nabla(Re(\pi))\|^2}$ (Gradient-Hamiltonian vector field)

Observation

- *Where defined, the flow ϕ_t takes $x \in X_s$ to $x' \in X_{s-t}$.*
- *Where defined, the flow ϕ_t preserves the symplectic structures on the X_s*

Idea of proof

- $S(A)$ finitely generated $\Rightarrow$ there exits toric degeneration of X to toric variety of Δ (Anderson 2010), called SAGBI or Gröbner degeneration
- Gradient-Hamiltonian flow extends to a continuous function $\phi : X \to X_0$, local diffeomorphism and symplectomorphism on a dense open U
- To prove continuity of ϕ one uses a version of *Łojasiewicz inequality* for algebraic sets
- Pull-back the toric integrable system on X_0 to X via ϕ

Simplest example

- X elliptic curve e.g. $X = \{y^2 = x^3 + 1\}$
- Degenerate X to a cuspidal cubic e.g. $X_c = \{y^2 = x^3 + c\}$
- Cuspidal cubic is singular toric variety e.g. $X_0 = \{y^2 = x^3\}$
- Gradient-Hamiltonian flow maps a topological torus to a pinched sphere

In progress

- Generalize to when $S(A)$ non-finitely generated
- (Karshon-Pabiniak) Would imply lower bounds for Gromov width of projective varieties
- In particular proves Gromov width of $X \geq 1$ (Biran's conjecture)

Thank you!

See:

- Harada, M.; Kaveh K. *Integrable systems, toric degenerations and Okounkov bodies.* arXiv.
- Anderson, D. *Okounkov bodies and toric degenerations.* Mathematische Annalen (2012).
- Nishinou, T.; Nohara, Y.; Ueda, K. *Toric degenerations of Gelfand-Cetlin systems and potential functions.* Adv. Math. 224 (2010).