# Toric degeneration and Newton-Okounkov body of Bott-Samelson variety

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# Outline

1. Why do we care NOBY?
2. Why do we care BS?
3. Construct a NOBY of BS!
4. Connections to Schubert varieties.
5. Connections to other toric degenerations

## $A \underset{\dashleftarrow}{\rightsquigarrow}$ Combinatorics

- Toric geometry: $A$- toric variety with a given Cartier divisor $D$ $\underset{\leftarrow}{\rightsquigarrow}$ Newton polytope $P_D$
- Tropical geometry: $A$- very affine variety $\underset{\dashleftarrow ?}{\rightsquigarrow}$ $\mathrm{Trop}(X)$, Tropical fan
- NOBY theory: $A$- $\begin{cases} \text{affine semigroup} \\ \text{algebra} \\ \text{linear series on variety} \end{cases}$ $\underset{\dashleftarrow ?}{\rightsquigarrow}$ (compact) convex set
  - Motivation came from the representation theory of reductive algebraic groups
  - Kaveh-Khovanskii and Lazarsfeld-Mustata developed the theory for cases without group-actions.
  - Applications:
    - (Kaveh-Khovanskii) polynomial growth of the Hilbert functions of a large class of graded algebras
    - (Kaveh-Khovanskii) A Brunn-Minkowski type inequality satisfied by the growth coefficients of these Hilbert functions.
    - (Lazarsfeld-Mustata, Kaveh-Khovanskii) Generalizations of Fujita approximation.
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# Definition

- $$S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n \longrightarrow \Delta(S) := \overline{\mathrm{Cone}(S)} \cap \pi_1^{-1}(m) \subset \mathbb{R}^{n+1}$$
  affine semigroup $\quad (m := [\mathbb{Z} : \pi_1(G(S))])$

  $L$, line bundle on a projective variety $X$ over $\mathbb{C}$, $\dim(X) = n$ $\rightarrow$ $A := \oplus_{m \geq 0} H^0(X, L^{\otimes m})$, graded $\mathbb{C}$-algebra

  $\downarrow$ conversion by $\nu$, $\mathbb{Z}^{n+1}$- valuation

  $S = S(X, L, \nu)$, affine semigroup

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  $\Delta(X, L, \nu) := \Delta(S)$,
  **Newton-Okounkov body (NOBY).**

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## Definition

Given a sequence of simple roots of $SL_n(\mathbb{C})$, $\mathbf{i} = (i_1, \ldots, i_n)$,

$$\mathrm{BS}(\mathbf{i}) := \prod_{j=1}^{n} P_{\alpha_{i_j}} / B^n,$$

$$(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \cdots, b_{n-1}^{-1} p_n b_n).$$

## Example

$$\mathbf{i} = (1, 2, 1) \text{ for } SL_3(\mathbb{C}),$$

$$\mathrm{BS}(\mathbf{i}) = P_{\alpha_1} \times P_{\alpha_2} \times P_{\alpha_1} / B^3$$

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- nonsingular projective varieties of dimension $n$
- an important tool in geometric representation theory.
- Originally defined as desingularizations of Schubert varieties (used to prove the rational singularities of Schubert varieties.)
- They have an action of a Borel subgroup.
- The projective coordinate ring of a BS splits into certain generalized Demazure modules.

# Idea

$L(\mathbf{m}) = \prod_{j=1}^{n} P_{i_j} \times_{B^n} \mathbf{k}^*_{m_1\omega_{i_1},\ldots,m_n\omega_{i_n}}$, line bundle on BS($\mathbf{i}$) $\longrightarrow$ toric degeneration $\mathcal{O}(D)$ line bundle on a nonsingula toric variety (Bott tower).

## Claim

*There is a valuation $\nu$ so that*

$$\Delta(BS(\mathbf{i}), L(\mathbf{m}), \nu) = P_D,$$

*the Newton polytope of a torus invariant divisor D, provided that D is base-point free.*

## Remark

The Newton polytope $P_D$ is the twisted cube of the Bott tower.

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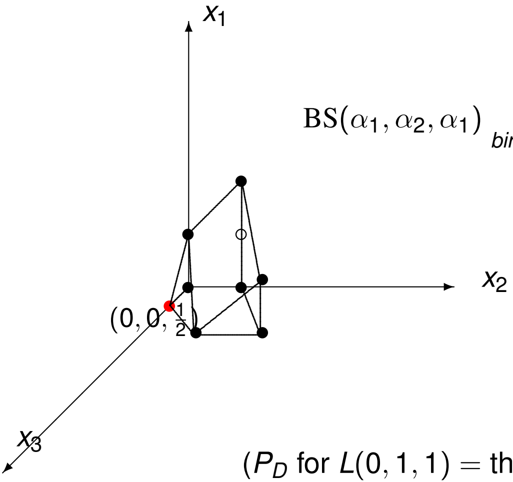
# Idea of proof

$$\mathcal{L} \to \mathcal{Z} \to \mathbb{C},$$

a “very nice” family (but need to be careful to choose a right valuation.)

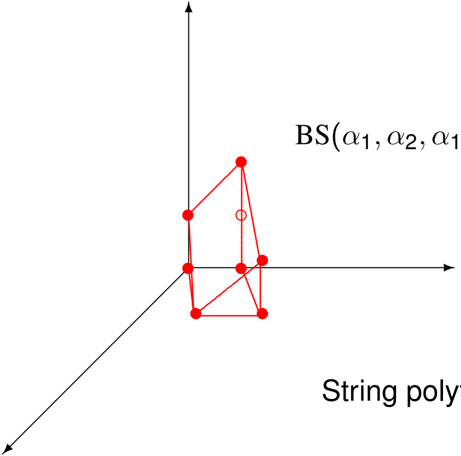
- Generalize Pasquier's toric degeneration by a deformation of the Borel group action:
$$\mathcal{Z} := \mathbb{C} \times \prod_{j=1}^{n} P_{\alpha_{i_j}} / B^n,$$
$$(t, p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) = (t, p_1 b_1, \lambda_t(b_1^{-1}) p_2 b_2, \cdots, \lambda_t(b_{n-1}^{-1}) p_n b_n)$$
Similarly for the line bundle $L(\mathbf{m})$.
- Prove the constancy of the dimensions $H^0(\mathcal{Z}_t, \mathcal{L}^{\otimes m}|_{\mathcal{Z}_t})$ using standard monomial theory and toric geometry.
- Embed $\mathbb{C}^{n+1}$, $(t, x_1, \ldots, x_n)$, nicely, giving a local coordinate system for each fiber.
- Valuation: For each fiber $\mathcal{Z}_t$, we take the lowest term valuation w.r.t $x_n > x_{n-1} > \cdots > x_1$.

$$\begin{array}{ccccccc} \mathcal{L}(m_n) & \hookrightarrow & \mathcal{L}(m_{n-1}, m_n) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{L}(m_1, \ldots, m_n) \\ \downarrow & & \downarrow & & \cdots & & \downarrow \\ \mathcal{Z}(i_n) & \hookrightarrow & \mathcal{Z}(i_{n-1}, i_n) & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{Z}(i_1, \ldots, i_n). \end{array}$$




$$\mathrm{BS}(\alpha_1, \alpha_2, \alpha_1) \underset{birational}{\rightarrow} X(s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}) = SL(3)/B$$

($P_D$ for $L(0, 1, 1)$ = the pull back of $L_{\alpha_1+\alpha_2}$; $D$ is not BF)



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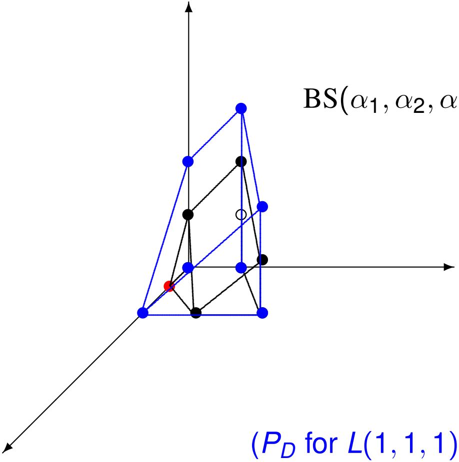
String polytope (Kaveh)



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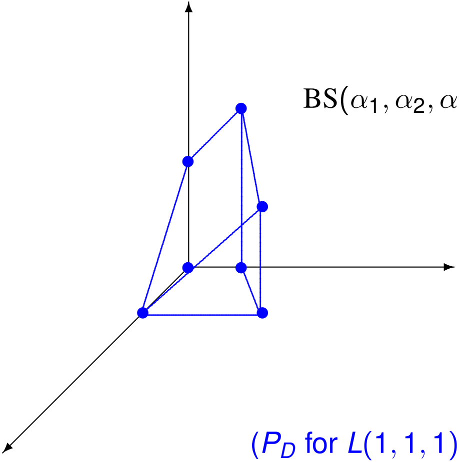
($P_D$ for $L(1,1,1)$; $D$ is BF but not ample)



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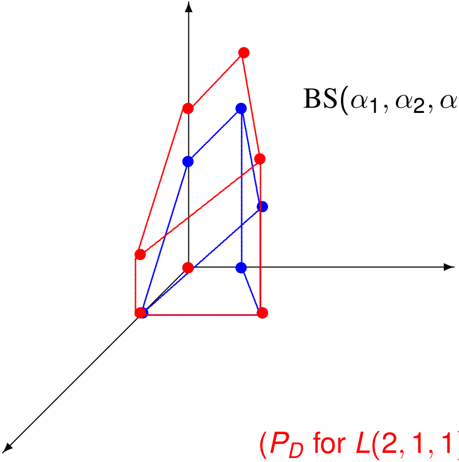

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($P_D$ for $L(2,1,1)$; $D$ is ample ($P_D$ is smooth)

- In fact, the proof shows more: the semigroup $S = S(\mathrm{BS}(\mathbf{i}), L(\mathbf{m}), \nu)$ is finitely generated by degree 1.
- Thus we can consider Anderson's toric degeneration.
- How are they related? - work in progress.