Combinatorics of simple polytopes, their Stanley–Reisner rings and moment-angle manifolds

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A moment-angle complex $\mathcal{Z}_K$

Definition

The *moment-angle complex* of a simplicial complex K on the vertex set $[m] = \{1, \ldots, m\}$ is a CW-complex $\mathcal{Z}_K = \bigcup_{I \in K} \left(\prod_{i \in I} \mathbb{D}^2 \times \prod_{i \notin I} \mathbb{S}^1 \right)$ (viewed as a subcomplex in a unit disk $(\mathbb{D}^2)^m \subset \mathbb{C}^m$).

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Examples: disks and joins

1) If $K = \Delta^n$ then $\mathcal{Z}_K = D^{2(n+1)}$

2) If $K = K_1 * K_2$ then $\mathcal{Z}_K = \mathcal{Z}_{K_1} \times \mathcal{Z}_{K_2}$.

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Examples: wedges of spheres

If $K = \{\{\varnothing\}, \{1\}, \{2\}, \{3\}\}$ for $m = 3$ then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres $S^3 \vee S^3 \vee S^3 \vee S^4 \vee S^4$.

A moment-angle manifold $\mathcal{Z}_P$

Simple polytopes

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Simple polytopes

We consider *simple convex n-dimensional polytopes P* in the Euclidean space $\mathbb{R}^n$ with scalar product $\langle\ ,\ \rangle$.
Such a polytope P can be defined as a bounded intersection of m halfspaces:

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i \geq 0 \quad \text{for } i = 1, \ldots, m\}, \qquad (*)$$

where $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. We assume that the hyperplanes defined by the equations $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle + b_i = 0$ are in general position, that is, at most n of them meet at a single point. We also assume that there are no redundant inequalities in $(*)$, that is, no inequality can be removed from $(*)$ without changing P.

Then P has exactly m *facets* given by

$$F_i = \{\mathbf{x} \in P \colon \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0\}, \quad \text{for } i = 1, \ldots, m.$$

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Let A_P be the $m \times n$ matrix of row vectors $\boldsymbol{a}_i$, and let $\boldsymbol{b}_P$ be the column vector of scalars $b_i \in \mathbb{R}$. Then we can write $(*)$ as

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon A_P \boldsymbol{x} + \boldsymbol{b}_P \geq \boldsymbol{0}\},$$

and consider the affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(\boldsymbol{x}) = A_P \boldsymbol{x} + \boldsymbol{b}_P.$$

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It embeds P into

$$\mathbb{R}^m_{\geq} = \{\boldsymbol{y} \in \mathbb{R}^m \colon y_i \geq 0 \quad \text{for } i = 1, \ldots, m\}.$$

A moment-angle manifold $\mathcal{Z}_P$

Definition

We define the space $\mathcal{Z}_P$ from the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geq} \end{array}$$

where $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$. The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{z \in \mathbb{C}^m \colon |z_i| = 1 \quad \text{for } i = 1, \dots, m\}$$

on $\mathbb{C}^m$. Therefore, $\mathbb{T}^m$ acts on $\mathcal{Z}_P$ with quotient P, and i_Z is a $\mathbb{T}^m$-equivariant embedding.

A moment-angle manifold $\mathcal{Z}_P$

By [3, Lemmae 7.2, 7.19], $\mathcal{Z}_P$ is a smooth closed 2-connected manifold of dimension $m+n$, called the *moment-angle manifold* corresponding to P.

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Examples: spheres and products

1) If $P = \Delta^n$ then $\mathcal{Z}_P = S^{2n+1}$
2) If $P = P_1 \times P_2$ then $\mathcal{Z}_P = \mathcal{Z}_{P_1} \times \mathcal{Z}_{P_2}$.

Examples

Truncation polytopes

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The corresponding moment-angle manifold $\mathcal{Z}_P$ is diffeomorphic to the connected sum of sphere products:

$$\mathop{\#}_{j=1}^{k} \left(S^{j+2} \times S^{2n+k-j-1}\right)^{\# j\binom{k+1}{j+1}},$$

where $X^{\# q}$ denotes a connected sum of q copies of X.

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Here the coefficients are the **Betti numbers** of a moment-angle manifold: $b_q(\mathcal{Z}_P) = (q-2)\binom{k+1}{q-1}$ for $3 \leq q \leq k+2$ and they do **not** depend on the dimension n of the truncation polytope P.

Examples

Generalized truncation polytopes

We call a *generalized truncation polytope* a k-vertex cut of a product of simplices $P = vc^k(\Delta^{n_1} \times \ldots \times \Delta^{n_r})$, for $n_1 \geq \ldots \geq n_r \geq 1, r \geq 1, k \geq 0$, which becomes a *truncation polytope* if $r = 1$ or $r = 2, n_2 = 1$.

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From the results of S.Gitler and S.Lopez de Medrano [5] and recently L.Chen, F.Fan and X.Wang [4] we know that for $r = 2$ the corresponding $\mathcal{Z}_P$ *is diffeomorphic to a connected sum of sphere products with* **two** *spheres in each product* .

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E.g. if $P = vc^1(\Delta^4 \times \Delta^3)$ then the resulting 17-dimensional smooth closed manifold $\mathcal{Z}_P$ is diffeomorphic to:

$$2S^3 \times S^{14} \# S^4 \times S^{13} \# S^7 \times S^{10} \# S^8 \times S^9.$$

Open questions

1) What is a criterion (in terms of combinatorial commutative algebra) for a simple polytope P to give as $\mathcal{Z}_P$ a connected sum of sphere products with **two** spheres in each product ?

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2) Is there any simple polytope P s.t. $\mathcal{Z}_P$ is topologically a **nontrivial** connected sum of sphere products including a product of **three** spheres ?

Now we are going to introduce a conjectural answer to the Question 1.

Fix a base field $\mathbb{k}$ and consider a simple n-dimensional polytope P with m facets. Denote by K_P the boundary ∂P^* of the dual simplicial polytope and let $\mathbb{k}[m] = \mathbb{k}[v_1, \ldots, v_m]$ be the graded polynomial algebra on m variables, $\deg(v_i) = 2$.

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Face ring

Let K be a simplicial complex on a set $[m] = \{1, \ldots, m\}$. The **face ring** of a simplicial complex K on $[m]$ is the quotient ring

$$\Bbbk[K] := \Bbbk[v_1, \ldots, v_m]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_k}$ for which $\{i_1, \ldots, i_k\}$ is **not** a simplex in K.

Simplicial complexes: necessary definitions and results

Fix a base field $\Bbbk$ and consider a simple n-dimensional polytope P with m facets. Denote by K_P the boundary ∂P^* of the dual simplicial polytope and let $\Bbbk[m] = \Bbbk[v_1, \ldots, v_m]$ be the graded polynomial algebra on m variables, $\deg(v_i) = 2$.

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Note that $\Bbbk[K]$ is a module over $\Bbbk[v_1, \ldots, v_m]$ via the quotient projection.

Golod property

A face ring $\Bbbk[K]$ is called **Golod** if the multiplication and all higher Massey operations in $\mathrm{Tor}_{\Bbbk[v_1,\ldots,v_m]}\big(\Bbbk[K],\Bbbk\big)$ are trivial. If so, the complex K is also called **Golod**.

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Theorem (L.'14)

Suppose $K = K_1 \cup_\sigma K_2$ is a simplicial complex obtained from two Golod complexes K_1 and K_2 by gluing along a common simplex. Then K is also Golod.

Minimally non-Golod property

If K itself is *not Golod* but *deleting any vertex* from K turns the restricted complex into a Golod one, then K is called **minimally non-Golod**.

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Polygon case

K_P is minimally non-Golod when P is an m-gon with $m \geq 4$.
K_P is Golod when P is a 3-gon.

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A simplicial complex K is called **flag** if any set of its vertices, pairwise joint by edges, forms a simplex in K.
E.g. K_P for an n-cube ($n \geq 2$) or an m-gon ($m \geq 4$) P but **not** a boundary of an n-simplex ($n \geq 2$).

Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'12)

If K is **flag** *then the following statements are equivalent:*
— $sk^1(K)$ is a chordal graph (that is, all induced cycles in a graph have length 3);
— $\mathcal{Z}_K$ has homotopy type of a wedge of spheres;
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Theorem (J.Grbić, T.Panov, S.Theriault, J.Wu'12)

If K_P is **flag** *then the following statements are equivalent:*
— P is an m-gon ($m \geq 4$);
— $\mathcal{Z}_P$ is homeomorphic to a connected sum of sphere products with two spheres in each product;
— K_P is a minimally non-Golod complex.

Conjecture

Z_P is homeomorphic to a **connected sum of sphere products** with 2 spheres in each product **if and only if** K_P is **minimally non-Golod** and **torsion free** (the latter means that all its induced subcomplexes have free integral homology groups).

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The "if" part firstly appeared in the work of J.Grbić, T.Panov, S.Theriault and J.Wu [6] as Question 3.5

Cohomology ring of $\mathcal{Z}_K$

The *Tor-groups* from the definition of a Golod complex acquire a topological interpretation by means of the following result on the cohomology of $\mathcal{Z}_K$:

Theorem (V.Buchstaber, T.Panov'98): first part

A cohomology algebra of a moment-angle complex $\mathcal{Z}_K$ is given by the isomorphisms

$$\begin{aligned} H^*(\mathcal{Z}_K;\Bbbk) &\cong \mathrm{Tor}_{\Bbbk[v_1,\dots,v_m]}(\Bbbk[K],\Bbbk) \\ &\cong H\big[\Lambda[u_1,\dots,u_m]\otimes\Bbbk[K],d\big] \\ &\cong \bigoplus_{I\subset[m]} \widetilde{H}^*(K_I;\Bbbk), \end{aligned}$$

$$\operatorname{bideg} u_i = (-1,2),\ \operatorname{bideg} v_i = (0,2);\quad du_i = v_i,\ dv_i = 0.$$

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Theorem (V.Buchstaber, T.Panov'98): second part

The last isomorphism is the sum of isomorphisms

$$H^p(\mathcal{Z}_K) \cong \sum_{I \subset [m]} \widetilde{H}^{p-|I|-1}(K_I),$$

and the ring structure is given by the maps

$$\widetilde{H}^{p-|I|-1}(K_I) \otimes \widetilde{H}^{q-|J|-1}(K_J) \to \widetilde{H}^{p+q-|I|-|J|-1}(K_{I\cup J}),$$

which are induced by the canonical simplicial maps $K_{I\cup J} \hookrightarrow K_I * K_J$ (join of simplicial complexes) for $I \cap J = \varnothing$ and **zero** *otherwise.*

Dual neighbourly polytopes

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2) duals to *cyclic polytopes* (the latter used in UBT for simplicial polytopes)

It was proved by F.Bosio and L.Meersseman [2] that boundary complexes of even dimensional dual neighbourly polytopes are **torsion free** when the dimension of the polytope P is **even**. The following statement describes the topology of $\mathcal{Z}_P$.

Dual neighbourly polytopes

Theorem (S.Gitler, S.Lopez de Medrano'12)

If P is an n-dimensional dual neighbourly polytope and n is even, then $\mathcal{Z}_P$ is diffeomorphic to a connected sum of sphere products with 2 spheres in each product **if and only if** *P is not a simplex.*

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Our conjecture relates this to the following result:

Theorem (L.'14)

If P is an n-dimensional dual neighbourly polytope and n is even, then K_P is minimally non-Golod **if and only if** *P is not a simplex.*

Generalized truncation polytopes

Theorem (A.Berglund, M.Jollenbeck'07)

For a truncation polytope P, its boundary complex K_P is minimally non-Golod **if and only if** *P is not a simplex.*

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Theorem (L.'13)

For $P = vc^k(\Delta^{n_1} \times \ldots \times \Delta^{n_r})$, $n_1 \geq \ldots \geq n_r \geq 1, r \geq 1, k \geq 0$, its boundary complex K_P is minimally non-Golod **if and only if** *$r = 1, 2$ and P is not a simplex.*

General vertex truncations

The next statement is likely to be the most general fact we know about $\mathcal{Z}_P$ topology change when a vertex truncation of a simple polytope P is performed:

Theorem (see [5] (for $k < 3n - m$) and [4] (general case))

If for a simple n-dimensional polytope P with m facets its moment-angle manifold $\mathcal{Z}_P$ is diffeomorphic to a connected sum of sphere products with 2 spheres in each product, then **the same is true** *for its k-vertex truncation $Q = vc^k(P)$ for any k.*

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Connected sums of polytopes and manifolds

Suppose $P = P_1 \# P_2$, i.e a connected sum of polytopes in 2 of their vertices, where P_1 and P_2 are n-dimensional simple polytopes.

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For their boundary complexes we have:

$$K = K_1 \underset{\sigma}{\#} K_2 = K_1 \cup_\sigma K_2 - \sigma,$$

where the 2 simplicial complexes are glued along a common maximal simplex.

Connected sums of polytopes and manifolds

Using the theorem on the Golodness of a gluing of 2 Golod complexes and the Buchstaber and Panov theorem we have the following general statement:

Theorem (L.'14)

If K_1 and K_2 are minimally non-Golod boundary complexes of simple n-dimensional polytopes with $n \geq 2$, then $K = K_1 \#_\sigma K_2$ is also minimally non-Golod.

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Open question

Is it true that if $\mathcal{Z}_{P_i}$ for $i = 1, 2$ is homeomorphic to a connected sum of sphere products with 2 spheres in each product, then the same holds for $\mathcal{Z}_P$ with $P = P_1 \# P_2$ (for all possible vertex truncations in the connected sum) ?

Suppose a simple n-dimensional polytope P has m facets.

Simple polytopes with few facets

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m=n+1

In this case P is a simplex, $\mathcal{Z}_P$ is a sphere and K_P is Golod.

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In this case P is a simplex, $\mathcal{Z}_P$ is a sphere and K_P is Golod.

m=n+2

In this case P is a product of 2 simplices, $\mathcal{Z}_P$ is a connected sum of sphere products with 2 spheres in each product and K_P is minimally non-Golod.

Simple polytopes with few facets

Suppose a simple n-dimensional polytope P has m facets.

m=n+1

In this case P is a simplex, $\mathcal{Z}_P$ is a sphere and K_P is Golod.

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In this case P is a product of 2 simplices, $\mathcal{Z}_P$ is a connected sum of sphere products with 2 spheres in each product and K_P is minimally non-Golod.

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In this case it follows from the results of S.Lopez de Medrano [7] that if P is **not** a product of 3 simplices then $\mathcal{Z}_P$ is a connected sum of sphere products with 2 spheres in each product.

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Theorem (L.'14) *For a simple n-dimensional polytope P with* $m = n + 3$ *facets its boundary complex* K_P *is minimally non-Golod if and only if P is* **not** *a product of 3 simplices.*

Example (m=n+4)

Consider $P = vc^1(\Delta^1 \times \Delta^1 \times \Delta^1)$. Using Buchstaber and Panov theorem it was pointed out by F.Bosio and L.Meersseman [2, Example 11.5] that $\mathcal{Z}_P$ is **not** homotopy equivalent to a connected sum of products of any number of spheres.

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Bigraded Betti numbers of simple polytopes

The dimensions of the bigraded components of the Tor-groups,

$$\beta^{-i,2j}(K) := \dim_{\Bbbk} \operatorname{Tor}^{-i,2j}_{\Bbbk[m]}(\Bbbk[K], \Bbbk), \quad 0 \le i,j \le m,$$

are known as the **bigraded Betti numbers** of $\Bbbk[K]$.
We denote $\beta^{-i,2j}(P) := \beta^{-i,2j}(K_P)$.

Cohomology ring of $\mathcal{Z}_P$

Cohomology of $\mathcal{Z}_P$ acquires a bigrading and the following statements hold.

Bigraded Poincaré duality (see [3, Theorem 8.18])

We have an equality:

$$\beta^{-i,2j}(P) = \beta^{-(m-n)+i,2(m-j)}(P).$$

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Betti numbers of $\mathcal{Z}_P$ via bigraded Betti numbers of P

We have an equality:

$$b^q(\mathcal{Z}_P) = \sum_{-i+2j=q} \beta^{-i,2j}(P).$$

Generalized truncation polytopes

The next result gives a complete calculation of the bigraded Betti numbers of all generalized truncation polytopes:

Theorem (L.'13)

Let $P = vc^k(\Delta^{n_1} \times \ldots \times \Delta^{n_r})$ for $n_1 \geq \ldots \geq n_r \geq 1, r \geq 1, k \geq 0$ be an arbitrary generalized truncation polytope. Denote by $A \geq 0$ the number of '1's in the set $\{n_1, \ldots, n_r\}$. Then the bigraded Betti numbers of P are given by the following formulae:
$(1 \leq i \leq k+r-1, 1 < l < d-1, d = n_1 + \ldots + n_r)$.

(a) $\beta^{-i,2(i+l)}(P) = \sum\limits_{\{n_{i_1},\ldots,n_{i_s}\}\subset\{n_1,\ldots,n_r\}:\, l=n_{i_1}+\ldots+n_{i_s}} \binom{k}{i-s}$.

(b) $\beta^{-i,2(i+1)}(P) = \beta^{-(k+r-i),2(d+k+r-i-1)}(P) = k\binom{k+r-1}{i} - \binom{k}{i+1} + A\binom{k}{i-1}$.

(c) $\beta^{0,0}(P) = \beta^{-(m-d),2m}(P) = 1$.

*The other bigraded Betti numbers are **zero** (we assume $\binom{b}{c} = 0$ if $b < c$ or one of them is negative).*

Generalized truncation polytopes

For truncation polytopes ($r = 1$) the bigraded Betti numbers were first calculated by T.Hibi and N.Terai in 1995.

For truncation polytopes ($r = 1$) the bigraded Betti numbers were first calculated by T.Hibi and N.Terai in 1995.
The crucial point of the proof is the following statement, which is obtained by Mayer–Vietoris exact sequence and induction on the number of vertex truncations:

Lemma

Let P be a generalized truncation polytope as above, K the boundary complex of the dual simplicial polytope, V the vertex set of K, and W a nonempty proper subset of V. Then

(i) $\widetilde{H}_i(K_W) = 0$ *for* $i \neq 0, n_{i_1} + \ldots + n_{i_s} - 1, d - 2$;

(ii) *Otherwise, for $i > 0$ the homology group $\widetilde{H}_i(K_W; \Bbbk)$ is nontrivial (and then $\cong \Bbbk$) if and only if W is a join of some $V(\Delta^{n_t}) \cup NV_t$, where NV_t, $1 \leq t \leq r$ is a subset of 'new' vertices (i.e, those corresponding to some of the k vertex truncations) of the dual simplicial polytope.*

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THANK YOU FOR YOUR ATTENTION!