Complex manifolds with maximal torus actions

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Topology of Torus Actions and Applications to Geometry and Combinatorics

8 August 2014, Friday

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Example

${}^{\exists}x \in M^G \Longrightarrow \dim G \leq \frac{1}{2} \dim M.$

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Proof.

$$\begin{array}{cccccc} ({}^{\forall}x \in M) & \dim G' & + & \dim G'_x & \leq & \dim M \\ & \vee\!| & & \vee\!| & & \\ ({}^{\exists}x \in M) & \dim G & + & \dim G_x & = & \dim M. \end{array}$$

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2. Non-equivariant version.

Example (Compact complex tori)

$M = \mathbb{C}^n/\Gamma$, Γ : lattice of $\mathbb{C}^n$.
$G = M = \mathbb{C}^n/\Gamma \cong (S^1)^{2n}$.

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$\Longrightarrow$
$(M, (S^1)^n)$ is equivariantly biholomorphic to a toric variety.

Example (Calabi-Eckmann manifolds)

Define $(\mathbb{C}^*)^m \curvearrowright (\mathbb{C}^k \setminus 0) \times (\mathbb{C}^{m-k} \setminus 0)$ to be

$$(g_1, \ldots, g_m) \cdot (z_1, \ldots, z_m) := (g_1 z_1, \ldots, g_m z_m).$$

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To each (M, G), we will assign 2 invariants

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"For $I \subset \{1, \ldots, k\}$, if $\bigcap_{i \in I} M_i \neq \emptyset$, then

$$\left\{ \sum_{i \in I} a_i \lambda_i \mid a_i \geq 0 \right\} \in \Delta".$$

2. a vector subspace $\mathfrak{h}$.
By compactness of M and that $G \curvearrowright M$ preserves the complex structure on M, we can define the complexified action

$$G^{\mathbb{C}} \curvearrowright M : \text{ holomorphic action.}$$

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The global stabilizers

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Its Lie algebra

$$\mathfrak{h} = \{u \otimes 1 + v \otimes \sqrt{-1} \mid u, v \in \mathfrak{g}, X_u + JX_v = 0\}$$

is a complex subspace of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

(X_u : fundamental vector field generated by $u \in \mathfrak{g}$)

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Proposition

$(\Delta, \mathfrak{h}, G)$ satisfies the following:

1. Δ is a nonsingular fan (w.r.t. the lattice $\mathrm{Hom}(S^1, G)$) in $\mathfrak{g}$,
2. for $p : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}$, $p|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$ is injective,
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1. $\longleftrightarrow$ effectivness of $G \curvearrowright M$ and Hausdorff-ness of M.
2. $\longleftrightarrow$ maximalness of $G \curvearrowright M$.
3. $\longleftrightarrow$ compactness and Hausdorff-ness of M/G.

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    3. $q : \mathfrak{g} \to \mathfrak{g}/p(\mathfrak{h})$ sends Δ to a complete fan in $\mathfrak{g}/p(\mathfrak{h})$.
- $\mathcal{F}_1 : \mathcal{C}_1 \to \mathcal{C}_2, \quad [M, G] \mapsto (\Delta, \mathfrak{h}, G)$.

For $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$,

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$G \curvearrowright X(\Delta)$ descends to $G \curvearrowright X(\Delta)/H$.

Proposition

$[X(\Delta)/H, G] \in \mathcal{C}_1$.

For $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$,

- $X(\Delta)$: toric variety associated with Δ
- $H := \exp(\mathfrak{h}) \subset G^{\mathbb{C}} \curvearrowright X(\Delta)$

Theorem (Battisti, 2012)

H acts on $X(\Delta)$ freely, holomorphically and properly. In particular, $X(\Delta)/H$ is a complex manifold. Moreover, $X(\Delta)/H$ is compact.

$G \curvearrowright X(\Delta)$ descends to $G \curvearrowright X(\Delta)/H$.

Proposition

$[X(\Delta)/H, G] \in \mathcal{C}_1$.

We have a map

$$\mathcal{F}_2 : \mathcal{C}_2 \to \mathcal{C}_1, \quad (\Delta, \mathfrak{h}, G) \mapsto [X(\Delta)/H, G].$$

Theorem

$\mathcal{F}_1$, $\mathcal{F}_2$ are inverses to each other.

Geometry and Topology of $[M, G] \in \mathcal{C}_1$ can be described in terms of $(\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$.

Recall

$$\{\text{toric variety}\} \xleftrightarrow{1-1} \{\text{rational fan}\}.$$

If we restrict our attention to "complete nonsingular" toric varieties and fans, the correspondence coincides with the correspondence between $\mathcal{C}_1$ and $\mathcal{C}_2$.

$\mathcal{C}_1$: complex manifolds with maximal torus actions

$\mathcal{C}_2$: triples $(\Delta, \mathfrak{h}, G)$

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- It can be shown that $\mathcal{F}_1 \circ \mathcal{F}_2 : \mathcal{C}_2 \to \mathcal{C}_2$ is the identity by the straightforward argument (because $X(\Delta)$ is concrete). The difficult part is that $\mathcal{F}_2 \circ \mathcal{F}_1 : \mathcal{C}_1 \to \mathcal{C}_1$ is the identity.

$$(M, G) \underset{?}{\cong} (X(\Delta)/H, G) \text{ for some } (\Delta, \mathfrak{h}, G) \in \mathcal{C}_2$$

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- ${}^{\exists}x \in M$ s.t. $\dim G + \dim G_x = \dim M$
$\Longrightarrow \operatorname{codim} G \cdot x = 2 \dim G_x$.
- $T_x M \cong T_x(G \cdot x) \oplus T_x M / T_x(G \cdot x)$ and hence

$$T_x(G \cdot x) = (T_x M)^{G_x}.$$

In particular, $G \cdot x$ is a complex submanifold.

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- By "sweeping" normal vector spaces by $G_x^{\mathbb{C}}$-action, we may show that

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- Define $(\Delta, \mathfrak{h})$ for the open subset $\bigcup_{G \cdot x} N(G \cdot x) \subseteq M$. Show the Proposition.

- $G^{\mathbb{C}} \to G^M$ induces a principal H-bundle

$$G^{\mathbb{C}} \times_{(G_x)^{\mathbb{C}}} (T_xM/T_x(G \cdot x)) \to G^M \times_{(G^M)_x} (T_xM/T_x(G \cdot x)).$$

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- $(M, G) \cong (X(\Delta)/H, G)$.

Definition

$(\Delta_1, \mathfrak{h}_1, G_1), (\Delta_2, \mathfrak{h}_2, G_2) \in \mathcal{C}_2$ are isomorphic

$\Longleftrightarrow$

${}^{\exists}\alpha : G_1 \to G_2$ isomorphism such that

1. $(d\alpha)_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ induces an isomorphism $\Delta_1 \to \Delta_2$,
2. $(d\alpha)_1 \otimes \mathrm{id}_{\mathbb{C}} : \mathfrak{g}_1^{\mathbb{C}} \to \mathfrak{g}_2^{\mathbb{C}}$ induces an isomorphism $\mathfrak{h}_1 \to \mathfrak{h}_2$.

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Theorem

For $[M_1, G_1], [M_2, G_2] \in \mathcal{C}_1$, TFAE :

1. (M_1, G_1) and (M_2, G_2) are weakly equivariantly biholomorphic.
2. M_1 and M_2 are biholomorphic.
3. $\mathcal{F}_1([M_1, G_1])$ and $\mathcal{F}_1([M_2, G_2])$ are isomorphic.

Thank you!