Torsions of cohomology of real toric manifolds

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Topology of torus actions and applications
to geometry and combinatorics
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Quasitoric manifolds and small covers

Definition (Davis-Januszkiewicz 1991)

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Definition

A toric manifold is a non-singular complete toric variety. A real toric manifold is the real locus of a toric manifold.

Quasitoric manifolds and small covers are topological analogues of toric manifolds and real toric manifolds respectively.

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  - ring over $\mathbb{Z}_2$ by Davis-Januszkiewicz (1991)
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Question

For a small cover M, would it be possible to describe $H^*(M;\mathbb{Z})$?

Torsion problems for small covers

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Is there a small cover M s.t. $H^*(M)$ contains an odd torsion?

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The following is still open:

Question (Torsion problem)

Is there a small cover M s.t. $H^*(M)$ contains a 4-torsion?

Building sets

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Definition

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$\mathcal{B}$: a set of nonempty subsets of $[n+1]$ such that

1. $\{1\}, \ldots, \{n+1\} \in \mathcal{B}$,
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Example

$\mathcal{B} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3,4\}, \{1,2,3,4\}\}$
$= \{1, 2, 3, 4, 12, 234, 1234\}$: Building set on $[4]$.

Graphical building sets

Definition

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$\mathcal{B}(G)$ is indeed a building set.

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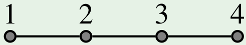
$G =$ 

$\mathcal{B}(G) = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$
13 or 124 are not elements of $\mathcal{B}(G)$.

Nestohedra and graph associahedra

- For $A, B \subset \mathbb{R}^n$, the Minkowski sum $A + B$ is

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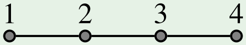
Theorem (Zelevinsky 2006)

Every nestohedron is a Delzant polytope. Furthermore, there is a natural way to obtain $P(\mathcal{B})$ by truncating the simplex.

Example

Associahedron As^3 (or Stasheff polytope)

$G = P_4 =$ 

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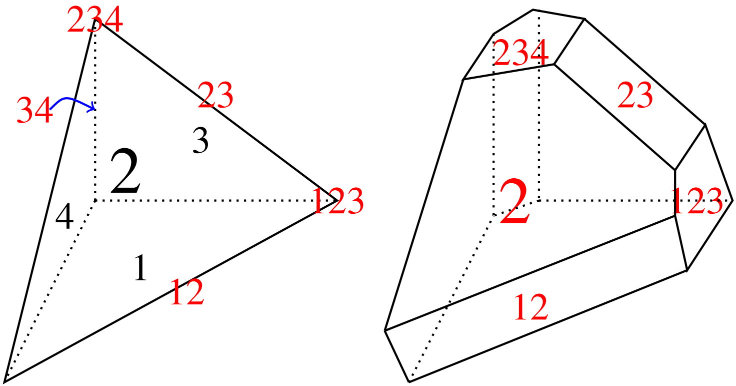

Torsion problem: Graphical case

- $P(\mathcal{B})$ is a Delzant polytope.
- $M(\mathcal{B})$: the real toric variety defined by $P(\mathcal{B})$ called the canonical real toric manifold of $\mathcal{B}$
- $M(\mathcal{B})$ is a small cover over $P(\mathcal{B})$.

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Theorem (Choi-P. 2013)

If $\mathcal{B} = \mathcal{B}(G)$ is graphical, then $H^*(M(\mathcal{B}); \mathbb{Z})$ has no odd torsion. Furthermore, the Betti numbers $\beta^i(M(\mathcal{B}))$ can be combinatorially described in terms of G.

Note: The number $\beta^i(M(\mathcal{B}(G))) =: a_i(G)$ is related to the a-number of G.

Building sets given by simplicial complexes

- K: any simplicial complex with vertex set $V(K)$

Definition

- A set $\sigma \subseteq V(K)$ is a minimal non-face if $\sigma \notin K$ but every proper subset of σ is a face of K. Denote by $MN(K)$ the set of such σ's.
- $\mathcal{B}(K)$ is the minimal building set such that $MN(K) \subseteq \mathcal{B}(K)$.

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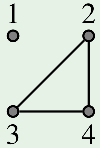

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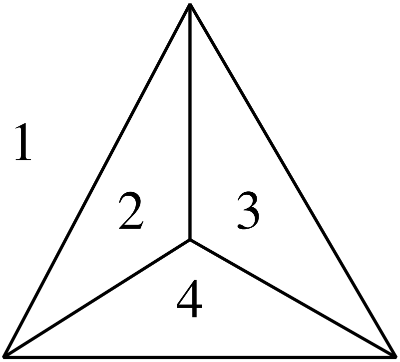

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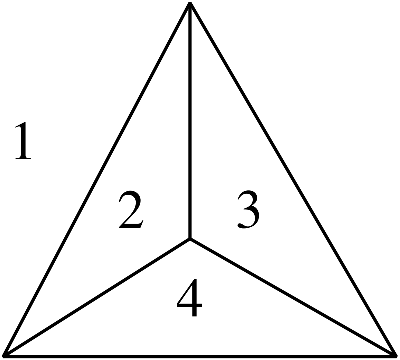

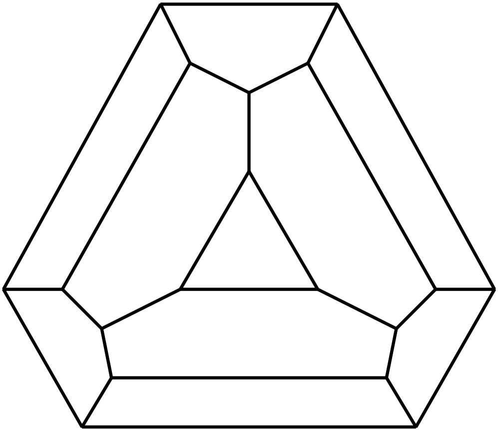

Note: This is a modified version of Bosio-Meersseman's construction.

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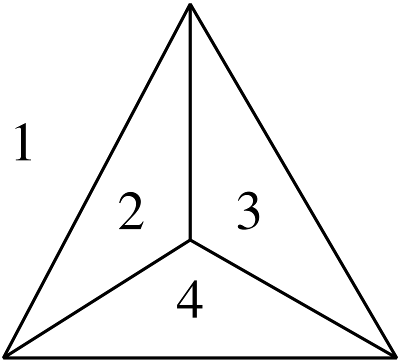

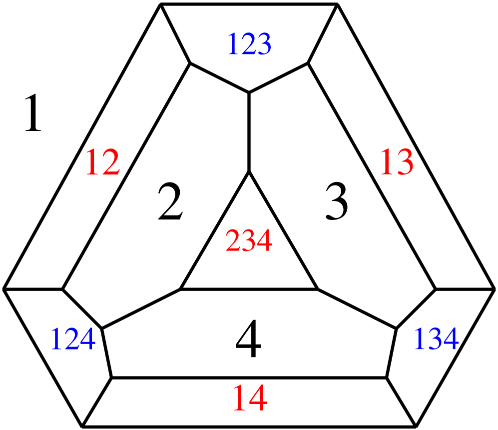

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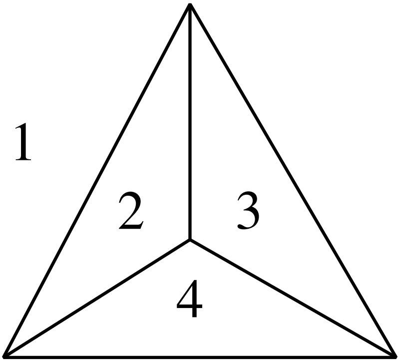

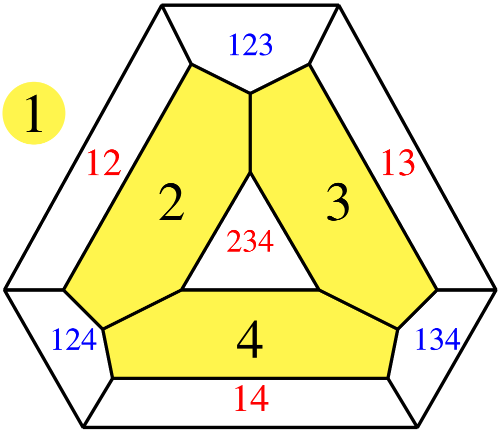

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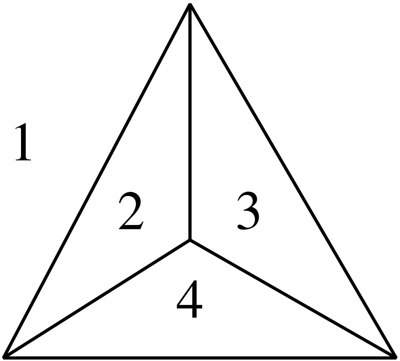

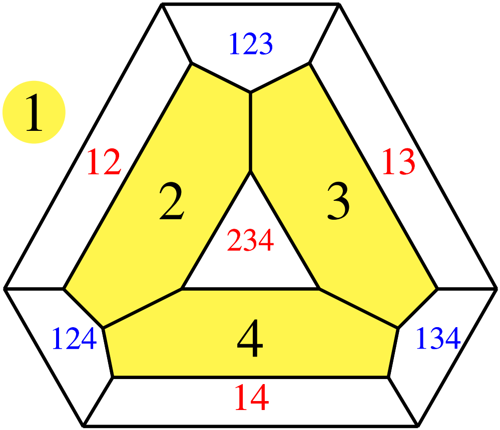

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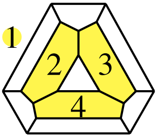
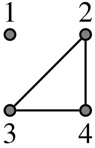
$P_\sigma =$ 

$\simeq$ 

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We have the Hochster-like formula for a small cover $M = M(P, \lambda)$.

- $\mathcal{F}$: the set of facets of P
- $S(\lambda) \subseteq 2^{\mathcal{F}}$: a set of subsets of $\mathcal{F}$ determined by λ
- $P_\sigma = \bigcup_{F \in \sigma} F$

Theorem (Suciu-Trevisan, Choi-P.)

$$H^*(M; G) = \bigoplus_{\sigma \in S(\lambda)} \tilde{H}^{*+1}(P_\sigma; G)$$

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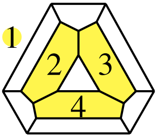
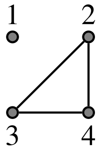
$P_\sigma =$  $\simeq$  $= K$

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When $M = M(\mathcal{B}(K))$, $P_\sigma \simeq K$ for some $\sigma \in \mathcal{F}$, but $\sigma \notin S(\lambda)$ in general.

Doublings of building sets and simplicial complexes

$\mathcal{B}'$: the building set obtained by "doubling" $\mathcal{B}$.

Example

$\mathcal{B}(K) = \{1, 2, 3, 4, 12, 13, 14, 234, 123, 124, 134, 1234\}$

$$\Longrightarrow (\mathcal{B}(K))' = \{1, 2, 3, 4, 1', 2', 3', 4', 11'22', 11'33', 11'44', 22'33'44', 11'22'33', 11'22'44', 11'33'44', 11'22'33'44'\}$$

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Remark

$(\mathcal{B}(K))' = \mathcal{B}(K')$ *when* $K' = \mathrm{Wed}_{(2,2,\ldots,2)} K$ *is known as the double of a simplicial complex* K. *It is obtained by multiple simplicial wedge operations from* K. *(Note: every minimal non-face is doubled.)*

Conclusion

Pick K so that $H^*(K)$ contains a q-torsion for odd q.

Theorem (Main result)

The small cover $M(\mathcal{B}(K'))$, which is also a real toric manifold, contains a q-torsion in its cohomology.

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Idea of proof:

- Since the double K' is a consecutive suspension of K, it still contains a q-torsion.
- If K' is a double, then the "yellow region"$= P_\sigma \simeq K'$ and $\sigma \in S(\lambda)$. Then the S-T-C-P formula

$$H^*(M;G) = \bigoplus_{\sigma \in S(\lambda)} \tilde{H}^{*+1}(P_\sigma;G)$$

and the UCT complete the proof.

Thank you for listening!