On diffeomorphic Moment-angle manifolds

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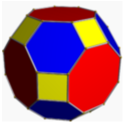
Outline

1. Introduction
   - Moment-angle manifolds
   - Statement of the problem
   - New notion
2. Examples and results
   - Cases of *Gr*-equivalence
   - Wedges
     - Case $n = d + 3$
     - The case of hexagon
   - Diff-equivalence is not *Gr*-equivalence: Toward a counterexample
3. Conclusion and open questions

Introduction

Notations



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We note d dimension of P, n the number of its facets. We call P a d-polytope or a (d, n)-polytope if we want to precise.

When needed, these facets are noted $F_1, \ldots, F_n$.

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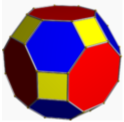

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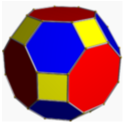

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Construction

There are several equivalent constructions of a moment-angle manifold. Let's recall one :
We consider the unit disc D^2 of $\mathbb{C}$, its boundary the unit circle S^1, and the unit polydisc $\Delta^n = (D^2)^n$ of $\mathbb{C}^n$. Then, Z_P if the subset of elements $(z_1, ..., z_n)$ of Δ^n such that:

$$\bigcap_{|z_i|<1} F_i \neq \emptyset$$

The manifold Z_P is compact and has dimension $d + n$.

Torus action

A moment-angle manifold is naturally equipped with an action of the torus $T^n = (S^1)^n$.

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Differential structure

The space constructed has a structure of a manifold with corners. It can be canonically smoothed to produce a differential manifold.

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Cohomology

One of the strongest tools to describe a space, and a manifold in particular, is cohomology.

In the case of moment-angle manifolds, there are several descriptions of the cohomology, usually in terms of the underlying polytope.

We get for instance:

Theorem

(Baskakov,) (Buchstaber, Panov)

$$H^*(Z_P, \mathbb{Z}) \simeq Tor_{\mathbb{Z}[X_{\mathcal{F}}]}(\mathbb{Z}(P), \mathbb{Z})$$

where $\mathbb{Z}(P)$ denotes the Stanley-Reisner ring of P (or P^*).

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Cohomology
Bigrading

For each k, $H^k(Z_P, \mathbb{Z})$ admits a natural decomposition into a direct sum:

$$H^k(Z_P, \mathbb{Z}) \simeq \bigoplus_{J \subset \mathcal{F}} \tilde{H}^{k-|J|-1}(F_J, \mathbb{Z})$$

Calling $H^{p,q}(Z_P, \mathbb{Z}) = \bigoplus_{\substack{J \subset \mathcal{F} \\ |J|=q}} \tilde{H}^p(F_J)$, this gives:

$$H^k(Z_P, \mathbb{Z}) \simeq \bigoplus_{p+q=k-1} H^{p,q}(Z_P, \mathbb{Z})$$

The dimensions of the spaces $H^{p,q}(Z_P, \mathbb{Z})$ are called the *bigraded betti numbers* of the polytope P, more precisely $\dim H^{p,q}(Z_P, \mathbb{Z})$ is usually referred as $b^{p-q+1,2q}$.

Problem

General question

Given two polytopes, when are the associated moment-angle manifolds "the same" or "similar"?

Different notions of similarity

Given two differential manifolds, we can find different notions of similarity, such as:
Isomorphic cohomology   same homotopy type
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Given two differential manifolds, we can find different notions of similarity, such as:

Isomorphic cohomology $\stackrel{?}{\Longrightarrow}$ same homotopy type $\stackrel{?}{\Longrightarrow}$ homeomorphism $\stackrel{?}{\Longrightarrow}$ diffeomorphism

Each reverse implication is false in general, but *open in the case of moment-angle manifolds*. Each one may give rise to an interesting investigation, but this won't be our purpose.

Similarity of moment-angle manifolds

In this lecture, we rather take the opposite direction. In fact, the particular structure of moment-angle manifolds allows to consider stronger notions of similarity than diffeomorphism.

Definition

Two polytopes P and Q are called *diff-equivalent* if Z_P and Z_Q are diffeomorphic.

Equivariantly diffeomorphic

A particularity of moment-angle manifolds is the existence of the recalled canonical action of T^n. We can ask if the respect of this action is a strong constraint.

Hence, the notion of (T^n)-equivariant diffeomorphism is too strong to produce an intersesting equivalence.
Nevertheless, demanding the respect of some subaction may be interesting, but we won't focus on that.

Equivariantly diffeomorphic

Theorem

(V. Buchstaber, T. Panov, 2002) ; (L. Meersseman, 2005)
Two moment-angle manifolds Z_P and Z_Q are equivariantly diffeomorphic if and only if the polytopes P and Q are the same (combinatorially equivalent).

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A new notion of similarity: The graded-equivalence

Another important property of moment-angle manifolds is the recalled bigrading of their cohomology. This leads to a new notion of similarity:

A new notion of similarity: The graded-equivalence

Definition

Consider two polytopes P and Q. We say that a diffeomorphism ϕ between Z_P and Z_Q is a *graded diffeomorphism* if it respects the bigrading of the homology groups, i.e. if ϕ^* sends $H^{p,q}(Z_Q)$ in $H^{p,q}(Z_P)$ for all p, q.
If there is a graded diffeomorphism between Z_P and Z_Q, then the polytopes P and Q are said *Gr-equivalent*.

Gr-equivalence and bigraded Betti numbers

We immediately remark that two *Gr*-equivalent polytopes have the same bigraded Betti numbers.

Case of connected sums

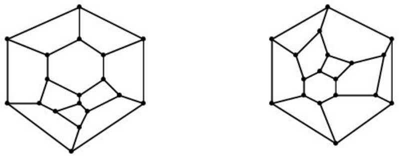
If Z_P and Z_Q are connected sums of 2-sphere products (i.e. where each term is the product of two spheres), and if the homology class of each sphere is "homogeneous", then P and Q are *Gr*-equivalent if and only if their bigraded Betti numbers coincide.

Gr-equivalence and bigraded Betti numbers

We immediately remark that two *Gr*-equivalent polytopes have the same bigraded Betti numbers.
The converse if false, even for diff-equivalence.

Counterexample

(Choi, 2012)
The following polytopes have the same bigraded Betti numbers, but nondiffeomorphic moment-angle manifolds:



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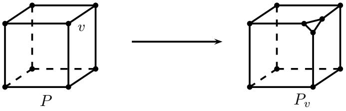
Examples and results

Vertex cutting I



Theorem

(B., L. Meersseman, 2005)
Let P a polytope, v and v' vertices of P. Then the polytopes P_v and $P_{v'}$ obtained by cutting v and v' respectively are Gr-equivalent.

The exact differential structure of Z_{P_v}, from the one of Z_P, has recently been settled :

Vertex cutting II

Theorem

(S. López de Medrano, S. Gitler, 2010) ; (L. Chen, F. Fan, X. Wang, 2014)
In the above context, we have :

$$Z_{P_v} \underset{diff}{\sim} \underbrace{\partial[D^2 \times (Z \setminus B^{n+d})]}_{canonic.\ smoothed} \#_{j=1}^{n-d} \binom{n-d}{j} S^{j+2} \times S^{n+d-j-1}$$

Dual neigbourly polytopes

Recall that if P is even-dimensional $d = 2d'$, then P is called dual neigbourly if any d' facets of P have nonempty intersection.

Theorem

(S. López de Medrano, S. Gitler, 2010)
Any two dual neigbourly polytopes with the same even dimension and number of facets are Gr-equivalent.

Connected sum

Construction

The connected sum (or blending) of two equidimensional simple polytopes consists in cutting a vertex to each one and gluing together the remainders (after a suitable projective transform) so that each facet containing the first vertex is assembled with another containing the second one.

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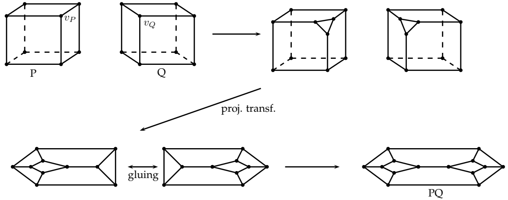


Connected sum

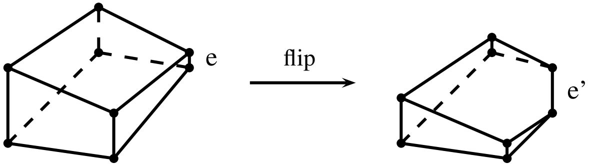
Gr-equivalence

The polytope $P\#Q$ obtained by gluing P and Q depends on the chosen vertices and the correspondance of facets.
Nevertheless, we have:

Theorem

Let P and Q two equidimensional polytopes. Then any two connected sums of P and Q are Gr-equivalent.

Flips

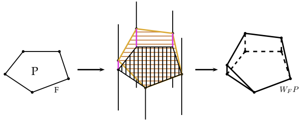


Theorem

Let P a polytope. Let Q_1 and Q_2 two polytopes obtained from P by a (p, q)-flip. Then, if the flipped simplices S_1 and S_2 have the same extremal facets, then Q_1 and Q_2 are Gr-equivalent.

Construction of wedges I

A wegde over some polytope P is given by the choice of some facet F. If P is realised in $\mathbb{R}^d$, we consider $P \times \mathbb{R}$ in $\mathbb{R}^{d+1}$ and two different nonvertical hyperplanes containing $F \times \{0\}$. The bounded component of their complement in $P \times \mathbb{R}$ is called the wedge over P on F (or relatively to F). It is combinatorially well defined, but depends on F. We note it $W_F P$.



This operation can be iterated and one can remark that it is in some sense "commutative and associative", in the sense that each facet of $W_F P$ (or iterated wedge) corresponds to some facet of P, that the wedge on a facet of $W_F P$ (or iterated

Construction of wedges II

wedge) only depends of the corresponding facet of P and the final wedge only depends on the number of times the wedging has been realised on each facet of P.
Thus, given a polytope P and a n-tuple of natural numbers $(m_1, ..., m_n)$, we can consider the wedge $W_{(m_1,...,m_n)}P$ obtained by repeating the wedging operation m_i times over the facet F_i for each i.

Case $n = d + 3$ I

Structure

We settle here the *Gr*-equivalence between simple $(d, d+3)$-polytopes. Let's recall the structure of such a polytope:

PROPOSITION

A polytope with three facets more than its dimension it obtained by iterated wedges over either the cube or the dual of a cyclic polytope $C_{2k,2k+3}$.

Also the structure of the corresponding moment-angle manifolds are known. If P is a wedge over a cube, it is the product of three simplices, hence Z_P is the product of three odd-dimensional spheres. In this case, no other polytope is diff-equivalent to it.

Case $n = d + 3$ II

Structure

Else we have:

Theorem

(S. López de Medrano, A. Verjovsky, 1996) Let P a wedge over $C^{*}_{2k,2k+3}$. *Then,* Z_P *is diffeomorphic to a connected sum of* 2*-sphere products, exactly* $2k+3$ *in number.*

Case $n = d + 3$ I

Numeric problem

Moreover the dimensions of the spheres are precised. Let's recall them.
Assume P is given by the $(2k+3)$-tuple $(m_1, ..., m_{2k+3})$ where each m_i is the number of wedgings on facet F_i, the facets being taken in the natural cyclic order.
For $1 \leq i \leq 2k+3$, we set $q_i = m_i + m_{i+2} + ... + m_{i+2k}$, the indices being taken modulo $2k+3$. Then for each i, we have a factor $S^{2k+2q_i+1} \times S^{2k+2S-2q_i+2}$ where S is the sum of all m_i's.
We then see that homology classes induced by sets of $k+1$ facets of $C^*_{2k,2k+3}$ and those induced by sets of $k+2$ facets have different parity. In particular, the decomposition of $H^k(Z_P)$ is trivial for each k. So in this case diff-equivalence implies *Gr*-equivalence.

Case $n = d + 3$ II

Numeric problem

We also see that the diffeomorphy class of Z_P is completely determined by the list of the values q_i.
We naturally can ask whether two $(2k + 3)$-tuples incude the same polytope. In fact, this is the case only if these $(2k + 3)$-tuples are in the same orbit of the natural action of the dihedral group D_{4k+6}.

Case $n = d + 3$

Transform

Theorem

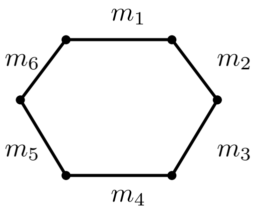
Two $(d, d+3)$-polytopes are Gr-equivalent if and only if the corresponding $(2k+3)$-tuples are joined in $\mathbb{Z}^{2k+3}$ by a chain of transforms that are, up to D_{4k+6}, given by:

$$(m_1, \dots, m_{2k+3}) \sim (m_1, m_2 - S_1, m_3, m_4, \dots, m_{2k}, m_{2k+1} + S_1)$$

where $S_1 = \sum_{i=2}^{2k+3} (-1)^i m_i$ is the alternate sum of the m_i's with m_1 excluded.

Hex-wedges
Construction

We consider here H the hexagon with facets (sides) ordered in a natural cyclic manner.
Wedging m_i times over facet $F_i, i = 1..6$ leads to a polytope $W_{(m_1,\ldots,m_6)}H$ we will call hex-wedge and note:



or

m_1

m_6 m_2

m_5 m_3

m_4

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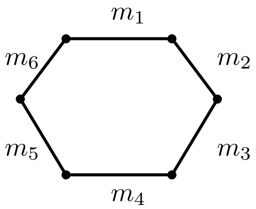

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Here too, the dihedral group D_{12} acts naturally on $\mathbb{N}^6$. We can easily show that two wedges are combinatorially equivalent if and only if the corresponding 6-tuples are in the same orbit. So, two such 6-tuples will be identified.

Hex-wedges

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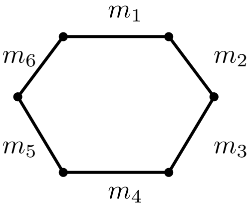

or

	m_1	
m_6		m_2
m_5		m_3
	m_4	

Definition

Two 6-tuples of natural numbers will be called equivalent (resp. *Gr*-equivalent) if the associated hex-wedges are diff-equivalent (resp. *Gr*-equivalent).

Hex-wedges I
Numeric problem

The moment-angle manifold over such a polytope is diffeomorphic to a connected sum of 2-products of spheres (S. López de Medrano, S. Gitler, 2010). The dimensions of these spheres can be computed :
Consider the following lists of integers:

- The list S_2 of the sums of two nonconsecutive m_i, i.e. $[m_1+m_3, m_1+m_4, m_1+m_5, m_2+m_4, m_2+m_5, m_2+m_6, m_3+m_5, m_3+m_6, m_4+m_6]$.
- The list S_3 of the sums of three nonconsecutive m_i, with $m_1+m_3+m_5$ and $m_2+m_4+m_6$ counted twice.
- The lists S_4 of the sums of four nonconsecutive m_i and S_4' with 1 added to each element of S_4.

Hex-wedges II
Numeric problem

We get a sphere of dimesion $(3+2k)$ for each k in S_2, a sphere of dimesion $4+2k$ for each k in S_3 and a sphere of dimesion $5+2k$ for each k in S_4 (remark that S_2 and S_4 induce odd-dimensional spheres and S_3 induce even-dimensional spheres).
Given two 6-tuples, they are Gr-equivalent if they have the same S_2, S_3 and S_4 (indeed they have the same S_2 if and only if they have the same S_4).
And they are diff-equivalent if they have the same S_3 and the same union $S_2 \cup S_4'$.

Hex-wedges

Gr-equivalent hex-wedges

Theorem

There are four families of transforms such that, given two polytopes obtained by hex-wedges, they are Gr-equivalent if and only if the systems $(m_1, ..., m_6)$ *and* $(m'_1, ..., m'_6)$ *are, up to the natural action of* D_{12}*, joined in* $\mathbb{Z}^6$ *by a chain of transforms belonging to these families.*

We give these families, and we can remark that they all have five parameters:

Hex-wedges
Transforms

I)
$$\begin{array}{ccc} & a & \\ a-\lambda & & a+\lambda \\ y & & x+\lambda \\ & b & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ a-\lambda & & a+\lambda \\ x & & y+\lambda \\ & b & \end{array}$$

We see here that some m_i is the mean of its two neigbours.

Hex-wedges

Transforms

II)
$$\begin{array}{ccc} & a & \\ c & & x \\ b & & y \\ & a & \end{array} \quad \text{and} \quad \begin{array}{ccc} & a & \\ c & & y \\ b & & x \\ & a & \end{array}$$

Here, two diametrically opposite m_is are equal.

Hex-wedges

Transforms

III)

$$\begin{array}{ccc} & x+\lambda & \\ b+\lambda & & a+\lambda \\ b & & a \\ & y & \end{array} \quad \text{and} \quad \begin{array}{ccc} & y+\lambda & \\ b+\lambda & & a+\lambda \\ b & & a \\ & x & \end{array}$$

Here, two sums of diametrically opposite m_is are equal.

Hex-wedges

Transforms

IV)

$$\begin{matrix} & a & \\ b & & a+x \\ b+y & & c+x \\ & c+y & \end{matrix} \quad \text{and} \quad \begin{matrix} & a & \\ b & & a+y \\ b+x & & c+y \\ & c+x & \end{matrix}$$

Here, the alternate sum of the m_is is zero.

Hex-wedges

Ungraded case

QUESTION

Does there exist two diff-equivalent hex-wedges that are not Gr-equivalent?

I have not the definite answer. Anyway, there are combinations that are close to counterexamples, for instance:

	6			7	
1		1	2		3
3		0	0		4
	4			-1	

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$$\begin{matrix} & 6 & \\ 1 & & 1 \\ 3 & & 0 \\ & 4 & \end{matrix} \qquad\qquad \begin{matrix} & 7 & \\ 2 & & 3 \\ 0 & & 4 \\ & -1 & \end{matrix}$$

Presentation

We show here evidence that there exists diff-equivalent polytopes that are not *Gr*-equivalent. Here too, we use wedges over neigbourly polytopes.

Indeed, we consider two even-dimensional neighbourly dual polytopes, with $n = d + 4$ and with a natural correspondance between their facets, so that one can pass from one to the other by four particular pairs of flips.

The two polytopes that are diff- but not *Gr*-equivalent are wedges over these neigbourly dual polytopes.

Requirement on flips

Assume $d = 2k$, $n = 2k + 4$. Let's note P_1 and P_2 our two neigbourly dual polytopes. Let's describe the structure of pairs of flips we require:
A pair of flips is made of a $(k, k+1)$-flip and a $(k+1, k)$-flip. To perform the pair of flips, the containing facets for the first flip must be the extremal facets for the second. Also, we require that for each pair of flips, exactly $k - 1$ among the other facets are extremal facets for the first flip and containing for the second. Hence, at each time, exactly one facet is remaining, i.e. is unmodified by either flip.

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For $1 \leq j \leq 4$, the containing facets of the jth first flip will be noted $A^j_1, ..., A^j_k$. The "common" facets, i.e. those that are extremal facets for the first flip and containing for the second will be noted $B^j_1, ..., B^j_{k-1}$. The other two extremal facets for the first flip are noted C^j_1, C^j_2, the two other containing facets for the second flip are noted D^j_1, D^j_2, and the remaining is noted E^j.

Homological changes produced by pairs of flips I

Indeed, when considering a $2k$-dimesional neigbourly dual polytope P, the homology of its moment-angle manifold is related to the sets of nonintersecting sets of $k+1$ facets of P. We can look how a pair of flips changes these sets. So consider a pair of flips with the prescribed properties. Only three sets of $k+1$ facets that met in P are globally disjoint after the flips, namely those containing all A_is and C_1 or C_2, and the set of containing facets for the second flip, which contains all B_is, D_1 and D_2. Conversely, there are three sets of $k+1$ facets that intersect after the flips and not before.
Consider a set $\mathcal{S}$ of $\mathcal{F}$ with $k+2$ facets. It induces homology in Z_P, thanks to the nonintersecting sets of $k+1$ facets it contains. For this homology to be increased (at least its dimension), $\mathcal{S}$

Homological changes produced by pairs of flips II

must contained (at least) one of the three abovementioned sets. But, by Poicaré duality, so must its complement.
There are indeed only three pairs of complementary sets of $k+2$ facets each containing one of the abovementioned sets of $k+1$ facets, one must contain all A_is, the other all B_is, D_1 and D_2, the first one two facets amongst C_1, C_2, E and the second the last one.
Recall that, after the repeated wedge, the degree of a homology class is increased by twice the sum of the number of performed wedge over the facets inducing it.

Values I

Finally we consider an integer N large enough, and we perform wedges over our polytopes. For a facet F, we note $m(F)$ the number of times the wedge has been performed over F (facets of the two polytopes are identified).
We set the following values:

$\sum_i m(A_i^1)$	$\sum_i m(B_i^1)$	$m(C_1^1)$	$m(C_2^1)$	$m(D_1^1)$	$m(D_2^1)$	$m(E^1)$
$7+N$	N	1	1	0	2	0
$\sum_i m(A_i^2)$	$\sum_i m(B_i^2)$	$m(C_1^2)$	$m(C_2^2)$	$m(D_1^2)$	$m(D_2^2)$	$m(E^2)$
$4+N$	$3+N$	1	2	0	0	1
$\sum_i m(A_i^3)$	$\sum_i m(B_i^3)$	$m(C_1^3)$	$m(C_2^3)$	$m(D_1^3)$	$m(D_2^3)$	$m(E^3)$
$5+N$	$4+N$	0	0	1	1	0
$\sum_i m(A_i^4)$	$\sum_i m(B_i^4)$	$m(C_1^4)$	$m(C_2^4)$	$m(D_1^4)$	$m(D_2^4)$	$m(E^4)$
$6+N$	N	0	2	1	1	1

Values II

These values give rise to two polytopes with the desired properties.
Warning: This construction does not really work with any polytope and flips with the described properties. A technical difficulty consists in finding sufficiently "random" flips to get compatibility for all these equations, for instance the facet C_1^1 cannot be the same as C_2^2 as $m(C_1^1) = 0$ and $m(C_2^2) = 1$.
Indeed, generically, the only compatibility equation concerns the total numer of wedgings performed, which is here equal to $11 + 2N$ for each flip.
Note also that, as N can be arbitrary chosen, there is no problem of sign for the $m(F)$.

Open questions

Gr-equivalence and *f*-vector

It is well known that the f-vector of a polytope (i.e. the number of faces of each dimension) does not determine its moment-angle manifold, not even its Betti numbers. Conversely, neither does the moment-angle manifold determine the f-vector when considering diff-equivalent polytopes. In my knowledge, there is even no proof that two diff-equivalent polytopes have the same dimension.

Question

Must Gr-equivalent polytopes have the same f-vector?

This can be shown to be true in the usual cases.

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Dual skeleton I

In the other sense, we might be interested in finding sufficient conditions for *Gr*-equivalence.
Classical question: Given a d-polytope P, which skeleton suffices to recover the complete combinatorial structure of P?

- For a general polytope, the $d-2$-skeleton is required.
- (R. Blind, P. Mani-Levitska, 1987)
  For a simple polytope, the 1-skeleton is sufficient so the comparsion of skeleta of P and Q is not relevant for our purpose.

Dual skeleton II

- (M. Perles)
  For a simplicial polytope, the knowledge of the $\lfloor \frac{d}{2} \rfloor = \lceil \frac{d-1}{2} \rceil$-skeleton, and no lower-dimensional, is sufficient to determine the polytope. In this sense, as one can recover a simple polytope from its dual, which is simplicial, one can also recover it from the $\lfloor \frac{d}{2} \rfloor$-skeleton of its dual.

We can ask if this bound can be improved if we only demand to determine polytopes up to *Gr*-equivalence. In the case of odd-dimensional polytopes, this bound cannot be improved.

CONJECTURE

Let P and Q two even-dimensional polytopes whose duals have the same $\frac{d}{2} - 1$-skeleton. Then P and Q are Gr-equivalent.

Thanks to public

Thank you for your attention.