Orbifold and non-orbifold linear symplectic quotients

Christopher Seaton, Rhodes College

joint work with Carla Farsi, Hans-Christian Herbig, Daniel Herden, and Gerald Schwarz

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Let $M_0 := Z/G$ denote the corresponding symplectic quotient.

When $G = \mathbb{T}^\ell$

When $G = \mathbb{T}^\ell$ is a torus, the action is described by the weight matrix $A = (a_{i,j}) \in \mathbb{Z}^{\ell \times n}$:

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Identifying $\mathfrak{g}^*$ with $\mathbb{R}^\ell$, the moment map is given by

$$J_i(z_1, \ldots, z_n) = \frac{1}{2} \sum_{j=1}^{n} a_{i,j} |z_i|^2, \qquad i = 1, \ldots, \ell.$$

Structures of the symplectic quotient

Smooth structure: $\mathcal{C}^\infty(M_0) := \mathcal{C}^\infty(V)^G/\mathcal{I}_Z^G$ where $\mathcal{I}_Z$ is the vanishing ideal of Z and $\mathcal{I}_Z^G := \mathcal{I}_Z \cap \mathcal{C}^\infty(V)^G$.

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(Schwarz 1976, Mather 1977) The smooth structure above coincides with the induced smooth structure as a subset of $\mathbb{R}^k$.

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The set $Z = J^{-1}(0)$ is usually not smooth, and G always fixes $0 \in Z$.

However, it is known that there are cases where M_0 is isomorphic to an orbifold (in terms of all of the above structures).

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The real invariants of the action are

$$p_1 = z_1\overline{z_1}, \quad p_2 = z_2\overline{z_2}, \quad p_3 = z_1 z_2, \quad p_4 = \overline{z_1}\,\overline{z_2}.$$

(in real coordinates $(z_1, z_2, \overline{z_1}, \overline{z_2})$, the weight matrix is $(-1, 1, 1, -1)$).

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The ideal $I_Z^G = \langle p_1 - p_2 \rangle$, so $\mathbb{R}[M_0] = \mathbb{R}[V]^G / I_Z^G$ is generated by the quadratics p_1, p_3, p_4 with relation $p_1^2 = p_3 p_4$.

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The symplectic quotient M_0 is homeomorphic to $\mathbb{C}/\mathbb{Z}_2$, and the above ring isomorphism can be lifted to an isomorphism $\mathcal{C}^\infty(M_0) \cong \mathcal{C}^\infty(\mathbb{C}/\mathbb{Z}_2)$ of Poisson algebras.

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(Sjamaar–Lerman, 1991) The Poisson algebra $\mathcal{C}^\infty(M_0)$ determines the stratification of M_0.

Orbifold cases for torus actions

Theorem (Farsi–Herbig–S., 2013)

Let $G = \mathbb{T}^\ell$ act on $\mathbb{C}^n$ with effective weight matrix $A \in \mathbb{Z}^{\ell \times n}$ of the form

$$A = \left(\begin{array}{cccc|cccc} a_{1,1} & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1,n-\ell} \\ 0 & a_{2,2} & & 0 & c_{21} & c_{22} & \cdots & c_{2,n-\ell} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{\ell,\ell} & c_{\ell,1} & c_{\ell,2} & \cdots & c_{\ell,n-\ell} \end{array}\right)$$

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with each $a_{i,i} < 0$ and each $c_{i,j} \geq 0$. Suppose exactly one $c_{i,j}$ in each row is positive. Then the symplectic quotient M_0 is graded regularly symplectomorphic to

$$\prod_{j=1}^{n-\ell} \mathbb{C}/\mathbb{Z}_{M_j}, \qquad M_j := lcm\{|a_{i,i}| : c_{i,j} \neq 0\} \left(1 + \sum_{i=1}^{\ell} \frac{c_{i,j}}{|a_{i,i}|}\right).$$

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(Wehlau, 1992) Any torus representation can be replaced by a stable torus representation without changing the GIT quotient.

Rational homology manifold condition for $\mathbb{T}^\ell$-quotients

Theorem (Farsi–Herbig–S., 2013)

Let $G = \mathbb{T}^\ell$. Suppose V is faithful and stable and $V^{\mathbb{T}^\ell} = \{0\}$. Then M_0 is a rational homology manifold if and only if, by permuting columns and row reducing over $\mathbb{Z}$, A can be put in the form $(D \mid C)$ where D is diagonal with strictly negative entries and C has nonnegative entries.

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This gives a necessary and sufficient condition for M_0 to be homeomorphic to an symplectic orbifold (and hence many non-orbifold examples).

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Example

The symplectic quotient associated to

$$A = \begin{pmatrix} -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is not a rational homology manifold and hence not homeomorphic to a symplectic orbifold.

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Theorem (Herbig–S.)

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Corollary

For stable unitary circle representations such that $V^{\mathbb{T}^1} = \{0\}$, M_0 is graded regularly symplectomorphic to a linear symplectic orbifold if and only if $\dim_{\mathbb{C}} V \leq 2$.

The Hilbert series

Definition

Let $R = \oplus_{k=0}^{\infty} R_k$ be a finitely generated graded algebra (over $\mathbb{K} = R_0$) with each R_k finite-dimensional. The **Hilbert series** of R is the formal series

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Hence

$$\mathrm{Hilb}_R(x) = \sum_{k=0}^{\infty} \gamma_k (1-x)^{k-\dim(R)}.$$

The Hilbert series of orbifold invariants

If Γ is a finite group and W is a unitary Γ-representation, for the Hilbert series of $\mathbb{C}[W]^\Gamma$ (complex invariants),

$$\gamma_0 = \frac{1}{|\Gamma|}$$

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$$\gamma_0 = \frac{1}{|\Gamma|}, \quad \text{and} \quad \gamma_1 = \frac{\#\text{pseudoreflections}}{2|\Gamma|}.$$

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$$\gamma_0 = \frac{1}{|\Gamma|}, \quad \gamma_1 = 0, \quad \text{and} \quad \gamma_2 = \gamma_3 = \frac{1}{12|\Gamma|} \sum_{i=1}^{r} |p_i|^2 - 1,$$

where the p_i are a "nice" collection of (complex) generating pseudoreflections.

The Hilbert series of a $\mathbb{T}^1$-symplectic quotient

Theorem (Herbig–S.)

Let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be the weight vector of an effective, stable $\mathbb{T}^1$-representation V with $V^{\mathbb{T}^1} = \{0\}$ and exactly one $a_i < 0$. Set $\boldsymbol{\alpha} = (|a_1|, \ldots, |a_n|) \in \mathbb{N}^n$, and $\boldsymbol{\alpha}_{\hat{j}} \in \mathbb{N}^{n-1}$ to be $\boldsymbol{\alpha}$ with the jth entry removed. For $\mathbb{R}[M_0]$,

$$\gamma_0(A) = \frac{s_{(n-2,n-2,n-3,\ldots,1,0)}(\boldsymbol{\alpha})}{s_{(n-1,n-2,\ldots,1,0)}(\boldsymbol{\alpha})}$$

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$$\gamma_0(A) = \frac{s_{(n-2,n-2,n-3,\dots,1,0)}(\boldsymbol{\alpha})}{s_{(n-1,n-2,\dots,1,0)}(\boldsymbol{\alpha})}, \qquad \gamma_1(A) = 0, \qquad \text{and}$$

$$\gamma_2(A) = \gamma_3(A) = \frac{\gamma_0(A)}{12} + \frac{s_{(n-3,n-3,n-3,n-4,\dots,1,0)}(\boldsymbol{\alpha})}{12\, s_{(n-1,n-2,\dots,1,0)}(\boldsymbol{\alpha})} S_{\boldsymbol{\alpha}}$$

$$+ \sum_{j=1}^{n} \frac{(\gcd(\boldsymbol{\alpha}_{\hat{j}})^2 - 1) s_{(n-3,n-3,n-4,\dots,1,0)}(\boldsymbol{\alpha}_{\hat{j}})}{12\, s_{(n-2,n-3,\dots,1,0)}(\boldsymbol{\alpha}_{\hat{j}})},$$

where s_λ denotes the Schur polynomial and $S_{\boldsymbol{\alpha}} = \sum_{i=1}^{n} \alpha_i^2$.

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When all of the weights are distinct,

$$\gamma_0(A) = \sum_{i=1}^{n} \frac{\alpha_i^{2n-3}}{\prod_{j\neq i} \alpha_i^2 - \alpha_j^2}$$

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$$-\sum_{i=1}^{n}\sum_{j\neq i:\gcd\{\alpha_k:k\neq j\}>1} \frac{\alpha_i^{2n-5}}{\prod_{k\neq i,j} \alpha_i^2 - \alpha_k^2}\left(\frac{1-\gcd\{\alpha_k : k\neq j\}^2}{12}\right).$$

The Hilbert series of a $\mathbb{T}^1$-symplectic quotient

Example

When $n = 3$, we have

$$\gamma_0(A) = \frac{\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3}{(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)},$$

and

$$\gamma_2(A) = \gamma_3(A) = \frac{\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}{12(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)} + \frac{\gcd(\alpha_1,\alpha_2)^2-1}{12(\alpha_1+\alpha_2)} + \frac{\gcd(\alpha_1,\alpha_3)^2-1}{12(\alpha_1+\alpha_3)} + \frac{\gcd(\alpha_2,\alpha_3)^2-1}{12(\alpha_2+\alpha_3)}$$

Proof method

Recall: *Let* $\mathbb{T}^1 \to \mathrm{U}(V)$ *be a stable unitary circle representation such that* $V^{\mathbb{T}^1} = \{0\}$*. If* $\dim_{\mathbb{C}} V \geq 3$*, then* M_0 *is not graded regularly symplectomorphic to a linear symplectic orbifold.*

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Such Γ are classified by duVal, and we can go through the list to see that their Hilbert series γ_i are always inconsistent with those of M_0 computed above.

Relations among the γ_i

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Gorenstein and Graded Gorenstein

(Stanley, 1978) A Noetherian graded domain R over $\mathbb{K} = R_0$ is **Gorenstein** if and only if it is Cohen-Macaulay and

$$H_R(1/x) = (-1)^{\dim R} x^{-a(R)} \operatorname{Hilb}_R(x)$$

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Proposition (Herbig–Herden–S.)

The graded Gorenstein condition

$$H_R(1/x) = (-1)^{\dim R} x^{\dim R} \operatorname{Hilb}_R(x)$$

is equivalent to the family of relations

$$\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \gamma_{m+k} = 0 \quad \text{for } m \geq 1.$$

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This appears to hold in the nonabelian case as well.

Ignoring the grading

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Let $\mathbb{T}^1 \to \mathrm{U}(V)$ *be a stable unitary circle representation such that* $V^{\mathbb{T}^1} = \{0\}$. *If* $\dim_{\mathbb{C}} V \geq 3$, *then* M_0 *is not regularly diffeomorphic to a linear symplectic orbifold.*

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Definition

Let V be a unitary G-representation. The action of $G_{\mathbb{C}}$ on V is **2-principal** if $V \smallsetminus V_{pr}$ has complex codimension at least 2, where V_{pr} denotes the preimage under the orbit map of the principal orbit type.

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Theorem (Herbig–Schwarz–S.)

Let G be a nontrivial compact connected Lie group and V a faithful unitary G-module. Assume the action of $G_{\mathbb{C}}$ on V is 2-principal and stable. Then M_0 is not symplectomorphic to a linear symplectic orbifold.

Example

The symplectic quotient M_0 associated to

$$A = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

is a rational homology manifold. M_0 is homeomorphic to $\mathbb{C}^2$.

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This representation is 2-principal and stable, so M_0 is not symplectomorphic to a linear symplectic orbifold.

Thank you!