Homology of manifolds with locally standard torus actions

Anton Ayzenberg
ayzenberga@gmail.com

Osaka City University

August 10, 2014
Topology of Torus Actions and Applications to Geometry and Combinatorics

Locally standard action

- $T^n = (S^1)^n$: compact torus;
- $T^n \curvearrowright \mathbb{C}^n$: standard representation (coordinate-wise rotations);
- M^{2n} : (smooth, compact, connected) manifold with an action of T^n. The action is called locally standard if it is locally modeled by the standard representation.

  This means there is an atlas $\{U_i\}$ on M such that $T^n \curvearrowright U_i$ is equivariantly diffeomorphic to $V_i \subset \mathbb{R}^{2n} = \mathbb{C}^n$ up to automorphism of a torus T^n.

Orbit space

Since $\mathbb{C}^n/T^n$ is identified with a nonnegative cone $\mathbb{R}^n_{\geqslant} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$, the orbit space M/T^n has the structure of n-dim. compact manifold with corners.

Orbit space

Since $\mathbb{C}^n/T^n$ is identified with a nonnegative cone $\mathbb{R}^n_{\geqslant} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$, the orbit space M/T^n has the structure of n-dim. compact manifold with corners.

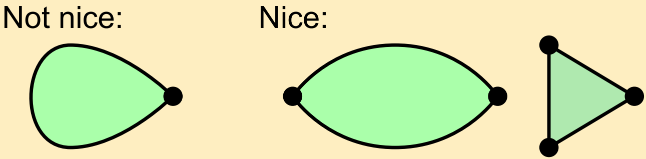
M.w.c. $Q = M/T$ has the following property: every face of Q of codimension k lies in exactly k distinct facets. Such m.w.c. are called nice. Only nice manifolds can appear as orbit spaces of locally standard actions.

Orbit space

Since $\mathbb{C}^n/T^n$ is identified with a nonnegative cone $\mathbb{R}^n_{\geqslant} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$, the orbit space M/T^n has the structure of n-dim. compact manifold with corners.

M.w.c. $Q = M/T$ has the following property: every face of Q of codimension k lies in exactly k distinct facets. Such m.w.c. are called nice. Only nice manifolds can appear as orbit spaces of locally standard actions.

Example



Most common situation

Suppose that M/T is acyclic and all its proper faces are acyclic. This situation is the most common in toric topology and toric geometry. Examples:

1. Complete smooth toric varieties (with induced action of $T^n \subset (\mathbb{C}^\times)^n$);
2. Quasitoric manifolds.

Most common situation

Suppose that M/T is acyclic and all its proper faces are acyclic. This situation is the most common in toric topology and toric geometry. Examples:

1. Complete smooth toric varieties (with induced action of $T^n \subset (\mathbb{C}^\times)^n$);
2. Quasitoric manifolds.

Let S be the simplicial poset dual to M/T.

E.g. underlying simplicial complex of a fan in the toric case, and simplicial complex dual to the polytope in quasitoric case.

Cohomology in acyclic case

Theorem (Masuda–Panov)

Suppose the m.w.c. M/T is acyclic and all its proper faces are acyclic. Then

$$H^*_T(M;\mathbb{Z}) \cong \mathbb{Z}[S],$$

— the face ring of the poset S, dual to M/T. Furthermore,

$$H^*(M;\mathbb{Z}) \cong \mathbb{Z}[S]/(l.s.o.p.),$$

where l.s.o.p. is a system of parameters of degree 2, determined by the characteristic map.

Remark

For examples above this result is widely known as Danilov–Jurkiewicz and Davis–Januszkiewicz theorems respectively.

General problem

Let M^{2n} be a compact manifold with locally standard action of T^n. Suppose we know the structure of the orbit space M/T, and the characteristic map. Are there simple descriptions of $H_T^*(M)$ and $H^*(M)$?

Explicit model

Let Q be a nice compact m.w.c. of dim n. Let $\rho\colon Y \to Q$ be a principal T^n-bundle over Q. Let λ be a "characteristic map" (to each facet $\mathcal{F}$ of Q this map associates a T^1-subbundle $\lambda(\mathcal{F})$ of $\rho|_{\mathcal{F}}$, and these subbundles should satisfy $(*)$-condition, similar to that of Davis–Januszkiewicz).

Explicit model

Let Q be a nice compact m.w.c. of dim n. Let $\rho\colon Y \to Q$ be a principal T^n-bundle over Q. Let λ be a "characteristic map" (to each facet $\mathcal{F}$ of Q this map associates a T^1-subbundle $\lambda(\mathcal{F})$ of $\rho|_{\mathcal{F}}$, and these subbundles should satisfy $(*)$-condition, similar to that of Davis–Januszkiewicz).

Consider the quotient space

$$X = Y/\sim$$

where

$$y_1 \sim y_2 \Leftrightarrow \begin{cases} \rho(y_1) = \rho(y_2) \in \bigcap_\alpha \mathcal{F}_\alpha, \\ y_2 \in \left(\prod_\alpha \lambda(\mathcal{F}_\alpha)\right) y_1. \end{cases}$$

The action of T on Y induces the action of T on X.

Explicit model

Example

If m.w.c. Q is a simple polytope, then every bundle $\rho\colon Y \to Q$ is trivial, and the space $X = (Q \times T)/\sim$ is a standard model for quasitoric manifold.

Explicit model

Example

If m.w.c. Q is a simple polytope, then every bundle $\rho\colon Y \to Q$ is trivial, and the space $X = (Q \times T)/\sim$ is a standard model for quasitoric manifold.

Theorem (Yoshida)

The space X is a compact manifold with locally standard torus action, and every manifold with l.s.t.a. is equivariantly homeomorphic to such model.

Thus, to compute the (equivariant) cohomology ring, it is sufficient to work with explicit models.

The setting for today's talk

As before, let Q be a nice manifold with corners, Y — a principal T-bundle over Q, and $X = Y/\sim$ — a manifold with locally standard action.

We assume that

1. All proper faces of Q are acyclic (Q itself may be arbitrary);
2. Y is a trivial T-bundle, i.e. $Y = Q \times T$, $X = (Q \times T)/\sim$.

This situation will be called "almost acyclic".

Simplicial posets

A finite partially ordered set (poset) S is called simplicial if it satisfies

1. $^{\exists}$ a minimal element $\hat{0} \in S$;
2. $^{\forall} I \in S$ the lower ideal $S_{\leqslant I} = \{J \in S \mid J \leqslant I\}$ is isomorphic to a boolean lattice $2^{[k]}$ ($=$ the face lattice of a simplex Δ^{k-1}). Elements of S are also called simplices. Number k is called the rank of I and is denoted $|I|$. Number $k-1$ is the dimension of simplex I.

Simplicial posets

A finite partially ordered set (poset) S is called simplicial if it satisfies

1. $^{\exists}$ a minimal element $\hat{0} \in S$;
2. $^{\forall} I \in S$ the lower ideal $S_{\leqslant I} = \{J \in S \mid J \leqslant I\}$ is isomorphic to a boolean lattice $2^{[k]}$ (= the face lattice of a simplex Δ^{k-1}). Elements of S are also called simplices. Number k is called the rank of I and is denoted $|I|$. Number $k-1$ is the dimension of simplex I.

Remark

Simplicial poset is a natural generalization of simplicial complex. One can treat simplicial poset geometrically as a face lattice of topological object (simplicial cell complex), made of simplices.

Simplicial posets

Most combinatorial (algebraical, topological) properties of simplicial complexes have their analogues for simplicial posets.

Simplicial posets

Most combinatorial (algebraical, topological) properties of simplicial complexes have their analogues for simplicial posets.

Let $I, J \in S$ be two simplices. Denote by $I \wedge J$ the set of meets (= greatest lower bounds, = intersections of simplices); and $I \vee J$ — the set of joins (= least upper bounds). Whenever $I \vee J$ is nonempty, the set $I \wedge J$ consists of a single element which is also denoted $I \wedge J$.

Face ring of simplicial poset

Let $\Bbbk$ be a field or $\mathbb{Z}$. To each simplex $I \in S$ we associate a formal variable v_I of degree $2|I|$. The graded ring

$$\Bbbk[S] = \frac{\Bbbk[v_I, I \in S]}{(v_I v_J - v_{I\wedge J}\sum_{L \in I \vee J} v_L; \quad v_{\hat{0}} - 1)}$$

is called the face ring of S. The sum over an empty set is assumed to be zero.

Simplicial posets from nice m.w.c.

If Q is a manifold with corners, then it has a natural stratification. The faces of Q form a partially ordered set with the order — reverse inclusion. We call this partially ordered set dual to Q and denote it S.

If Q is a nice manifold with corners, then S is a simplicial poset.

Recall the result of Masuda and Panov:

Theorem (Masuda–Panov)

Let M^{2n} be a manifold with locally standard action of T^n. Suppose the m.w.c. M/T is acyclic and all its proper faces are acyclic, and let S be simplicial poset dual to M/T. Then

$$H_T^*(M;\mathbb{Z}) \cong \mathbb{Z}[S].$$

This result explains the appearance of face rings in toric topology.

Recall the result of Masuda and Panov:

Theorem (Masuda–Panov)

Let M^{2n} be a manifold with locally standard action of T^n. Suppose the m.w.c. M/T is acyclic and all its proper faces are acyclic, and let S be simplicial poset dual to M/T. Then

$$H_T^*(M;\mathbb{Z}) \cong \mathbb{Z}[S].$$

This result explains the appearance of face rings in toric topology. In the almost acyclic case we have

Theorem (A.–Masuda–Park–Zeng)

Let $X = (Q \times T)/\sim$ be a manifold with l.s.a. and suppose the orbit space Q has acyclic proper faces. Let S be the simplicial poset dual to Q. Then

$$H_T^*(X,\mathbb{Z}) \cong H^*(Q;\mathbb{Z}) \oplus \mathbb{Z}[S]$$

— the direct sum of graded rings.

Ordinary cohomology

The (non-equivariant) cohomology ring $H^*(X)$, even in almost acyclic case, is more complicated.

Ordinary cohomology

The (non-equivariant) cohomology ring $H^*(X)$, even in almost acyclic case, is more complicated.

Remark

Consider the Borel construction $X \times_T ET$ and the standard fibration $\pi\colon X \times_T ET \to BT$. Let $i\colon X \hookrightarrow X \times_T ET$ be an inclusion of a fiber. Then we have two ring homomorphisms

$$H^*(BT) \xrightarrow{\pi^*} H_T^*(X) \xrightarrow{i^*} H^*(X)$$

which compose to zero in positive degrees. Thus there is a ring homomorphism

$$\bar{i}\colon H_T^*(X)/(\pi^* H^{>0}(BT)) \to H^*(X)$$

Difficulty

$$\bar{i}\colon H_T^*(X)/(\pi^* H^{>0}(BT)) \to H^*(X)$$

When the orbit space Q and all its faces are acyclic, this homomorphism is an isomorphism. The image $\pi^* H^2(BT)$ gives $(l.s.o.p.)$ in $H_T^*(X) \cong \mathbb{Z}[S]$.

This $l.s.o.p.$ is mentioned in theorems of [DJ], [DJ], and [MP].

Difficulty

$$\bar{i}\colon H_T^*(X)/(\pi^* H^{>0}(BT)) \to H^*(X)$$

When the orbit space Q and all its faces are acyclic, this homomorphism is an isomorphism. The image $\pi^* H^2(BT)$ gives $(l.s.o.p.)$ in $H_T^*(X) \cong \mathbb{Z}[S]$.

This *l.s.o.p.* is mentioned in theorems of [DJ], [DJ], and [MP].

Warning!

In general (even in almost acyclic case), $\bar{i}$ is neither injective, nor surjective.

A few more definitions from combinatorial algebra

The homology of a simplicial poset S is the homology of its geometrical realization.

Let $I \in S$ be a simplex. The link of I is a poset

$$\operatorname{lk}_S I = \{J \in S \mid J \geqslant I\},$$

with induced order. It is again a simplicial poset.

A few more definitions from combinatorial algebra

Definition

A simplicial poset S is called Cohen–Macaulay, if $\widetilde{H}_i(\mathrm{lk}_S I) = 0$ for each $I \in S$ and $i < \dim \mathrm{lk}\, I$. If the same property holds for all $I \neq \hat{0}$, then S is called Buchsbaum.

Triangulations of spheres are typical examples of CM complexes.

Triangulations of manifolds are typical examples of Buchsbaum complexes.

Face numbers

Let $f_i(S)$ be the number of i-dimensional simplices in a pure simplicial poset S of dimension $n-1$. As usual, define h-numbers by the relation

$$\sum_{i=0}^{n} h_i(S)t^{n-i} = \sum_{i=0}^{n} f_{i-1}(S)(t-1)^{n-i}.$$

Define h'-numbers by

$$h'_i(S) = h_i(S) + \binom{n}{i}\left(\sum_{j=1}^{i-1}(-1)^{i-j-1}\widetilde{b}_{j-1}(S)\right) \text{ for } 0 \leqslant i \leqslant n;$$

where $\widetilde{b}_{j-1}(S) = \dim \widetilde{H}_{j-1}(S)$.

Relation to face ring

Two theorems motivate the importance of h- and h'-numbers in commutative algebra.

Relation to face ring

Two theorems motivate the importance of h- and h'-numbers in commutative algebra.

Theorem (Stanley)

If S is Cohen–Macaulay, and $(l.s.o.p.)$ is a system of parameters of degree 2 in $\Bbbk[S]$ (thus regular sequence), then the dimension of degree $2i$ component of $\Bbbk[S]/(l.s.o.p.)$ is $h_i(S)$.

This is well known.

Relation to face ring

Two theorems motivate the importance of h- and h'-numbers in commutative algebra.

Theorem (Stanley)

If S is Cohen–Macaulay, and $(l.s.o.p.)$ is a system of parameters of degree 2 in $\Bbbk[S]$ (thus regular sequence), then the dimension of degree $2i$ component of $\Bbbk[S]/(l.s.o.p.)$ is $h_i(S)$.

This is well known.

Theorem (Schenzel, Novik–Swartz)

If S is Buchsbaum, and $(l.s.o.p.)$ is a system of parameters of degree 2 in $\Bbbk[S]$, then the dimension of degree $2i$ component of $\Bbbk[S]/(l.s.o.p.)$ is $h'_i(S)$.

Returning to locally standard actions

Why do we need this algebra?

Returning to locally standard actions

Why do we need this algebra?

Observation 1

If Q is a manifold with corners, which is acyclic together with all faces, then its dual poset S is Cohen–Macaulay.

Returning to locally standard actions

Why do we need this algebra?

Observation 1

If Q is a manifold with corners, which is acyclic together with all faces, then its dual poset S is Cohen–Macaulay.

Observation 2

If Q is a manifold with corners, in which all proper faces are acyclic, then the dual poset S is Buchsbaum. Moreover, S has the same homology as ∂Q.

Additive structure of $H^*(X)$

As before, let Q be a m.w.c. with acyclic proper faces; $X = (Q \times T)/\sim$ — a manifold with l.s.t.a.; and S — simplicial poset dual to Q.

Also assume Q is orientable, and, consequently, so is X.

Let $\delta^i \colon H^i(\partial Q) \to H^{i+1}(Q, \partial Q)$ be the connecting homomorphism in the cohomology exact sequence for the pair $(Q, \partial Q)$.

Under this assumptions and notation, there holds...

Theorem

There exists a bigraded structure in the cohomology of X:

$$H^k(X) = \bigoplus_{i+j=k} H^{i,j}(X),$$

where

$$H^{i,j}(X) \cong \begin{cases} H^i(Q) \otimes H^j(T), & \text{if } i > j; \\ H^i(Q, \partial Q) \otimes H^j(T), & \text{if } i < j; \end{cases}$$

and

$$\dim H^{i,i}(X) = \begin{cases} h'_i(S) + \binom{n}{i} \dim H^i(Q) - \binom{n}{i} \dim \ker \delta^{i-1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } i \geqslant 2; \\ h'_1(S) + n \dim H^1(Q), \ \text{if } i = 1; \\ 1, \ \text{if } i = 0; \end{cases}$$

Over a field, there holds bigraded Poincare duality $H^{i,j}(X) \cong H^{n-i,n-j}(X)$.

Outline of proof

There is an obvious stratification of m.w.c. Q by dimensions of faces:

$$Q_0 \subset Q_1 \subset \ldots \subset Q_{n-1} = \partial Q \subset Q$$

By multiplying with T we get a filtration on $Y = Q \times T$:

$$Y_0 \subset Y_1 \subset \ldots \subset Y_{n-1} \subset Y$$

The quotient map $Y \to X = Y/\sim$ induces the filtration on X:

$$X_0 \subset X_1 \subset \ldots \subset X_{n-1} \subset X,$$

which coincides with the filtration of X by orbit types.

These filtrations generate homology spectral sequences ${}^{Q}E^{r}_{*,*}$, ${}^{Y}E^{r}_{*,*}$ and ${}^{X}E^{r}_{*,*}$ respectively.

Outline of proof

The spectral sequence ${}^{Q}E^{r}_{*,*} \Rightarrow H^{*}(Q)$ has a simple ⌉-shaped structure, if all proper faces of Q are acyclic. All higher differentials can be described explicitly.

Obviously, ${}^{Y}E^{r}_{*,*} \cong {}^{Q}E^{r}_{*,*} \otimes H^{*}(T)$. Thus there is also a complete description of ${}^{Y}E^{r}_{*,*}$.

We have a morphism of spectral sequences $f^{r}_{*}\colon {}^{X}E^{r}_{*,*} \to {}^{Y}E^{r}_{*,*}$, induced by the filtration-preserving map $f\colon Y \to X = Y/\sim$.

Basic theorem

The map $f^{2}_{*}\colon {}^{Y}E^{2}_{p,q} \to {}^{X}E^{2}_{p,q}$ is an isomorphism if $q < p$, and injective if $q = p$.

Outline of proof

Basic theorem

The map $f_*^2\colon {}^XE^2_{p,q} \to {}^YE^2_{p,q}$ is an isomorphism if $q < p$, and injective if $q = p$.

Since ${}^XE^r_{p,q} = 0$ for $q > p$ by dimensional reasons, this theorem allows to describe the whole spectral sequence ${}^XE$.

The bigraded structure in $H_*(X)$ (and cohomology) comes from the bigraded structure in $Y = Q \times T$, namely $H_{i,j}(Y) \cong H_i(Q) \otimes H_j(T)$.

The proof of Basic theorem relies on the homological algebra and makes use of the version of Verdier duality between cellular sheaves and cosheaves.

Thank you!

- A. Ayzenberg, *Homology of torus spaces with acyclic proper faces of the orbit space*, arXiv:1405.4672.
- Justin M. Curry, *Sheaves, Cosheaves and Applications*, arXiv:1303.3255v1
- Mikiya Masuda, Taras Panov, *On the cohomology of torus manifolds*, Osaka J. Math. 43 (2006), 711–746.
- Isabella Novik, Ed Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math., 222 (2009), 2059–2084.
- Peter Schenzel, *On the Number of Faces of Simplicial Complexes and the Purity of Frobenius*, Math. Zeitschrift 178 (1981), 125–142.
- Takahiko Yoshida, *Local torus actions modeled on the standard representation*, Adv. Math., 227 (2011), 1914–1955.