Hamiltonian loops on symplectic blow ups

Andrés Pedroza

Universidad de Colima

The set up and the problems

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- Understand the homotopy type of $\mathrm{Ham}(M, \omega)$.
- Determine $\pi_1(\mathrm{Ham}(M, \omega))$.

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- $\mathrm{Ham}(S^2 \times S^2, \omega \oplus \lambda\omega)$ for $\lambda \geq 1$. (Gromov, McDuff, Abreu, Lalonde, etc.)
- $\mathrm{Ham}(\mathbb{C}P^2, \omega_{FS}) \simeq PU(3)$. (Gromov)

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- $\mathrm{Flux} : \pi_1(\mathrm{Symp}(M, \omega)) \to H^*(M; \mathbb{R})$
- $\mathcal{S} : \pi_1(\mathrm{Ham}(M, \omega)) \to QH^*(M, \Lambda)$
- $\mathcal{A} : \pi_1(\mathrm{Ham}(M, \omega)) \to \mathbb{R}/\mathcal{P}(M, \omega)$.

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- Flux : $\pi_1(\mathrm{Symp}(M,\omega)) \to H^*(M;\mathbb{R})$
- $\mathcal{S}$: $\pi_1(\mathrm{Ham}(M,\omega)) \to QH^*(M,\Lambda)$
- $\mathcal{A}$: $\pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}/\mathcal{P}(M,\omega)$.

Our aim is to show that some loops in $\mathrm{Ham}(M,\omega)$ induced nontrivial loops in $\mathrm{Ham}(\widetilde{M},\widetilde{\omega}_\rho)$.

$(\widetilde{M},\widetilde{\omega}_\rho)$: is the one point blow up of weight ρ of (M,ω).

Example: Complex projective space

Consider $(\mathbb{C}P^n, \omega_{FS})$ for $n \geq 2$ and the Hamiltonian circle action

$$e^{2\pi it} \cdot [z_0 : \cdots : z_n] = [z_0 : e^{2\pi it} z_1 : \cdots : e^{2\pi int} z_n]$$

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- $x_0 = [1 : 0 \cdots : 0]$ is fixed by the action.
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- Blow up $(\mathbb{C}P^n, \omega_{FS})$ at x_0 to get $(\widetilde{\mathbb{C}P}^n, \widetilde{\omega}_\rho)$
- The loop ψ induces a Hamiltonian loop $\widetilde{\psi}$ on the blow up

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In our example

- $\mathcal{P}(\mathbb{C}P^n, \omega_{FS}) = \mathbb{Z}\pi$
- $\mathcal{P}(\widetilde{\mathbb{C}P}^n, \widetilde{\omega}_\rho) = \mathbb{Z}\pi + \mathbb{Z}\pi\rho^2$

Weinsteins morphism

For our initial loop ψ in

$$\mathrm{Ham}(\mathbb{C}P^n, \omega_{FS})$$

we have

$$\mathcal{A}[\psi] = \left[-\frac{n}{2}\pi\right] \in \mathbb{R}/\mathbb{Z}\langle\pi\rangle$$

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and in the blow up, $\widetilde{\psi} \in \mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho)$

$$\mathcal{A}(\widetilde{\psi}) = \left[-\frac{n}{2}\pi + \frac{\pi\rho^{2n}}{2(1-\rho^{2n})}\left(\frac{\rho^2}{(n-1)!} - n\right)\right] \in \mathbb{R}/\mathbb{Z}\langle\pi, \pi\rho^2\rangle.$$

Results

Theorem (P.)

Let ψ *be the Hamiltonian loop defined above and* $0 < \rho < 1$*. Then* ψ *induces a loop* $\widetilde{\psi}$ *in* $\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho)$ *and*

- $\widetilde{\psi}$ *has finite order in* $\pi_1(\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ *if* ρ^2 *is rational;*
- $\widetilde{\psi}$ *has infinite order in* $\pi_1(\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ *if* ρ^2 *is transcendental.*

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Let ψ be the Hamiltonian loop defined above and $0 < \rho < 1$. Then ψ induces a loop $\widetilde{\psi}$ in $\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho)$ and

- *$\widetilde{\psi}$ has finite order in $\pi_1(\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ if ρ^2 is rational;*
- *$\widetilde{\psi}$ has infinite order in $\pi_1(\mathrm{Ham}(\widetilde{\mathbb{C}P^n}, \widetilde{\omega}_\rho))$ if ρ^2 is transcendental.*

Theorem (P.)

Let (M, ω) be a closed symplectic manifold and $(\widetilde{M}, \widetilde{\omega}_\rho)$ the blow up at $x_0 \in M$ of weight ρ. If $\psi \in \pi_1(\mathrm{Ham}(M, \omega))$ has a representative that can be lifted to a loop $\widetilde{\psi}$ in $\mathrm{Ham}(\widetilde{M}, \widetilde{\omega}_\rho)$, then

$$\mathcal{A}_{\widetilde{M}}(\widetilde{\psi}) = \left[\mathcal{A}_M(\psi) + \int_0^1 c_\rho(M, \omega, H_t)dt\right]$$

in $\mathbb{R}/\mathcal{P}(\widetilde{M}, \widetilde{\omega}_\rho)$ where H_t is the normalized Hamiltonian function of the loop ψ.