Monomial equivariant embeddings of quasitoric manifolds and the problem of existence of invariant almost complex structures.

Andrey Kustarev

joint work with V. M. Buchstaber,
Steklov Mathematical Institute

Topology of torus actions and applications to geometry and combinatorics, 7 Aug 2014, Daejong.

Quasitoric manifolds

Our main object is a quasitoric manifold M associated with combinatorial data (P, Λ).

Quasitoric manifolds

Our main object is a quasitoric manifold M associated with combinatorial data (P, Λ).

We will provide all necessary definitions following [Buchstaber-Panov-Ray, 2007].

Quasitoric manifolds

Our main object is a quasitoric manifold M associated with combinatorial data (P, Λ).

We will provide all necessary definitions following [Buchstaber-Panov-Ray, 2007].

In short, M will be constructed as a smooth orbit space $\mathcal{Z}_P/K$, where $\mathcal{Z}_P \subset \mathbb{C}^m$ is a T^m-invariant real-algebraic submanifold and $K \subset T^m$ is a toric subgroup acting on $\mathcal{Z}_P$ freely.

Simple polytopes

Consider simple n-polytope P with m facets, determined by the system of linear inequalities

$$P = \{x \in \mathbb{R}^n \colon \ \langle a_i, x\rangle + b_i \geqslant 0\}, \ i = 1 \ldots m,$$

none of which is redundant.

Simple polytopes

Consider simple n-polytope P with m facets, determined by the system of linear inequalities

$$P = \{x \in \mathbb{R}^n \colon \; \langle a_i, x \rangle + b_i \geqslant 0\},\; i = 1 \ldots m,$$

none of which is redundant.

Equivalent matrix form is

$$A_P x + b_P \geqslant 0.$$

Simple polytopes

There exists a natural affine map

$$i_P : \mathbb{R}^n \to \mathbb{R}^m : \quad i_P(x) = A_P x + b_P.$$

Simple polytopes

There exists a natural affine map

$$i_P\colon \mathbb{R}^n \to \mathbb{R}^m: \quad i_P(x) = A_P x + b_P.$$

Then

$$i_P(P) = (\mathbb{R}^m_{\geqslant 0}) \cap i_P(\mathbb{R}^n),$$

where $\mathbb{R}^m_{\geqslant 0}$ consists of points in $\mathbb{R}^m$ with all coordinates being non-negative.

Moment-angle manifolds

Denote by

$$\rho\colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant 0}$$

the standart moment map $z \to |z|^2$.

Moment-angle manifolds

Denote by

$$\rho \colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant 0}$$

the standart moment map $z \to |z|^2$.

We may now define a moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^m$ as a pullback in the following diagram:

$$\begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ {\scriptstyle \rho_P}\downarrow & & \downarrow{\scriptstyle \rho} \\ P & \xrightarrow{i_P} & \mathbb{R}^m_{\geqslant 0} \end{array}$$

Moment-angle manifolds

Theorem [Buchstaber-Panov-Ray, 2007]. The manifold $\mathcal{Z}_P$ is a nonsingular intersection of $(m-n)$ real hypersurfaces in $\mathbb{C}^m$ of the form

$$\sum_{k=1}^{m} c_{j,k}(|z_k|^2 - b_k) = 0, \, j = 1 \ldots (m-n).$$

Moment-angle manifolds

Theorem [Buchstaber-Panov-Ray, 2007]. The manifold $\mathcal{Z}_P$ is a nonsingular intersection of $(m-n)$ real hypersurfaces in $\mathbb{C}^m$ of the form

$$\sum_{k=1}^{m} c_{j,k}(|z_k|^2 - b_k) = 0,\ j = 1 \dots (m-n).$$

Therefore, $\mathcal{Z}_P$ is a smooth real algebraic T^m-invariant $(m+n)$-dimensional submanifold in $\mathbb{C}^m$.

Combinatorial data

Recall that facets of P are enumerated from 1 to m.

We will say that set of indices $I \subset [1, m]$ is *admissible* iff the intersection of facets

$$F_I = \bigcap_{i \in I} F_i$$

is not empty in P.

Combinatorial data

Denote by $T_I \subset T^m$ the coordinate torus

$$T_I = \prod_{i \in I} T_i,$$

where T_j is j-th coordinate subgroup in $T^m = T_1 \times \ldots \times T_m$.

Combinatorial data

Consider homomorphism

$$\lambda\colon T^m \to T^n$$

that satisfies *independency condition:*

$$F_I \text{ is face of } P \implies \lambda \text{ is a monomorphism on } T_I.$$

Combinatorial data

The homomorphism λ is uniquely determined by integer $(n \times m)$-matrix Λ such that

$$v \text{ is a vertex } \Longrightarrow \det \Lambda_v = \pm 1.$$

Here matrix Λ_v is formed by columns of Λ with indices from set I, where $v = \bigcap_{i \in I} F_i$.

Quasitoric manifolds

Consider short exact sequence

$$1 \longrightarrow K \longrightarrow T^m \xrightarrow{\lambda} T^n \longrightarrow 1,$$

where $K \subset T^m$ is a subgroup isomorphic to torus T^{m-n}.

Quasitoric manifolds

Consider short exact sequence

$$1 \longrightarrow K \longrightarrow T^m \xrightarrow{\lambda} T^n \longrightarrow 1,$$

where $K \subset T^m$ is a subgroup isomorphic to torus T^{m-n}.

By construction, K acts on moment-angle manifold $\mathcal{Z}_P$ freely.

Quasitoric manifolds

We will call smooth $2n$-dimensional manifold

$$M = \mathcal{Z}_P / K$$

a *quasitoric manifold, associated with combinatorial data* (P, Λ).

Quasitoric manifolds

We will call smooth $2n$-dimensional manifold

$$M = \mathcal{Z}_P / K$$

a *quasitoric manifold, associated with combinatorial data* (P, Λ).

Denote by $\pi\colon M \to P$ the corresponding projection map.

Motivation: toric varieties

Any *projective non-singular toric variety* X has the structure of quasitoric manifold.

Motivation: toric varieties

Any *projective non-singular toric variety* X has the structure of quasitoric manifold.

The action of compact torus is Hamiltonian and P in this case is the image of moment map.

Motivation: toric varieties

Any *projective non-singular toric variety* X has the structure of quasitoric manifold.

The action of compact torus is Hamiltonian and P in this case is the image of moment map.

If $\mathfrak{F}$ is a complete non-singular fan determining X, then in this case matrix Λ is formed by the set of vectors corresponding to 1-dimensional cones of $\mathfrak{F}$.

Isotropy subgroups

Denote by $S(T^n)$ the set of all subgroups in T^n.

There is a natural *characteristic function* $\chi\colon M \to S(T^n)$ defined as

$$\chi(x) = \text{ isotropy subgroup of } x.$$

Isotropy subgroups

Since T^n is abelian, χ is well-defined on orbit space $P = M/T^n$.

Isotropy subgroups

Since T^n is abelian, χ is well-defined on orbit space $P = M/T^n$.

For a given face $F \subset P$ denote by $\overset{\circ}{F}$ the set of points of F not belonging to any face of smaller dimension.

Isotropy subgroups

Since T^n is abelian, χ is well-defined on orbit space $P = M/T^n$.

For a given face $F \subset P$ denote by $\overset{\circ}{F}$ the set of points of F not belonging to any face of smaller dimension.

Then χ is constant on each open face $\overset{\circ}{F}_I \subset P$ and $\chi(\overset{\circ}{F}_I)$ is equal to subgroup $\lambda(T_I) \subset T^n$.

Codimension one isotropy subgroups

Denote by q the number of pairwise distinct subgroups $H_i \in \mathrm{Im}(\chi) \subset S(T^n)$ that are $(n-1)$-dimensional.

Codimension one isotropy subgroups

Denote by q the number of pairwise distinct subgroups $H_i \in \mathrm{Im}(\chi) \subset S(T^n)$ that are $(n-1)$-dimensional.

If $H_i = \chi(R)$ for some $R \subset P$, then necessarily R is an edge of polytope P.

Codimension one isotropy subgroups

The number q of distinct codimension one isotropy subgroups is an invariant of quasitoric manifold M. It can not exceed the number of edges of P.

Codimension one isotropy subgroups

The number q of distinct codimension one isotropy subgroups is an invariant of quasitoric manifold M. It can not exceed the number of edges of P.

For products of quasitoric manifolds (and not only for them), q is significantly smaller than number of edges of $P = M/T^n$.

Codimension one isotropy subgroups

Each subgroup H_i determines a homomorphism

$$T^n \longrightarrow (T^n/H_i) = S^1,$$

once orientation in quotient group T^n/H_i is fixed.

Codimension one isotropy subgroups

Each subgroup H_i determines a homomorphism

$$T^n \longrightarrow (T^n/H_i) = S^1,$$

once orientation in quotient group T^n/H_i is fixed.

We will denote by v_{H_i} the integer n-vector corresponding to representation $T^n \to (T^n/H_i)$.

The main representation

We may associate with each pair (P, Λ) the representation

$$\Phi\colon T^n \to T^q$$

consisting of one-dimensional representations $T^n \to (T^n/H_i)$, $i \in [1, q]$.

Denote by W the corresponding integer $(n \times q)$-matrix formed by vectors v_{H_i}.

The main representation

Our main result is that Φ generates an *equivariant embedding* of M to linear Euclidean space $\mathbb{R}^n \times \mathbb{C}^q$.

Motivation: Mostow-Pale theorem

Non-equivariant case: by Whitney theorem, any compact closed n-dimensional manifold M can be embedded into $\mathbb{R}^{2n}$.

Motivation: Mostow-Pale theorem

Non-equivariant case: by Whitney theorem, any compact closed n-dimensional manifold M can be embedded into $\mathbb{R}^{2n}$.

There is a well-developed theory of embeddings and immersions of smooth manifolds, based on classical theory of characteristic classes.

Motivation: Mostow-Pale theorem

By Mostow-Pale theorem, any compact manifold M with smooth action of Lie group G may be equivariantly embedded into $\mathbb{R}^N$ w.r.t. some representation $G \to GL(N)$.

Motivation: Mostow-Pale theorem

By Mostow-Pale theorem, any compact manifold M with smooth action of Lie group G may be equivariantly embedded into $\mathbb{R}^N$ w.r.t. some representation $G \to GL(N)$.

Problem. How can one estimate a dimension of equivariant embedding?

Monomial maps

Each integer m-vector $a = (a_i)$ determines a monomial real-algebraic map $\varphi_a \colon \mathbb{C}^m \to \mathbb{C}$ by the formula:

$$\varphi_a((z_1, \ldots, z_m)) = \prod_{i=1}^{m} \hat{z}_i^{|a_i|},$$

where

- $\hat{z}_i = z_i \quad$ if $a_i > 0$.
- $\hat{z}_i = 1 \quad$ if $a_i = 0$.
- $\hat{z}_i = \bar{z}_i \quad$ if $a_i < 0$.

Monomial maps

Lemma. Monomial map

$$\varphi_a|_{\mathcal{Z}_P} : \mathcal{Z}_P \to \mathbb{C}$$

factors through M if and only if

$$a \in \Lambda^*(\mathbb{Z}^n) \subset \mathbb{Z}^m.$$

Monomial embedding of M

Denote by $\varphi_M\colon \mathcal{Z}_P \to \mathbb{C}^q$ the smooth map whose components are monomials determined by columns of $(m \times q)$-matrix $\Lambda^T W$.

Monomial embedding of M

Denote by $\varphi_M\colon \mathcal{Z}_P \to \mathbb{C}^q$ the smooth map whose components are monomials determined by columns of $(m \times q)$-matrix $\Lambda^T W$.

Recall that standart map $\rho_P\colon \mathcal{Z}_P \to P$ is a projection onto orbit space $\mathcal{Z}_P \to \mathcal{Z}_P/T^m$.

Monomial embedding of M

Theorem 1. The map

$$(\rho_P \times \varphi_M)\colon \mathcal{Z}_P \longrightarrow (P \times \mathbb{C}^q)$$

induces a T^n-equivariant embedding of $M = \mathcal{Z}_P / K$ corresponding to linear representation $\mathrm{Id} \times \Phi$.

Sketch of the proof

We don't use Mostow-Pale theorem, but follow the similar scheme.

1. We construct an equivariant monomial map

$$\tilde{\varphi}_M \colon M \longrightarrow \bigoplus_{F \subset P} \mathbb{C}^{\dim F},$$

   and show that $\rho_P \times \tilde{\varphi}_M$ distinguishes points of M.

2. Components corresponding to edges of P are enough to generate all other components of this map. So $\tilde{\varphi}_M$ collapses into φ_M.

Example: $M = \mathbb{C}P^1$, $P = \Delta^1$.

$$\mathcal{Z}_P = S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2,$$
$$M = \mathcal{Z}_P / \{(t,t)\} = \mathbb{C}P^1.$$

We have $n = 1$, $q = 1$, and

$$(\rho_P \times \varphi_M)((z_1, z_2)) = (|z_1|^2,\, z_1 \bar{z}_2).$$

Example: $M = \mathbb{C}P^1$, $P = \Delta^1$.

$$\mathcal{Z}_P = S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2,$$

$$M = \mathcal{Z}_P / \{(t, t)\} = \mathbb{C}P^1.$$

We have $n = 1$, $q = 1$, and

$$(\rho_P \times \varphi_M)((z_1, z_2)) = (|z_1|^2, \; z_1\bar{z}_2).$$

So $\mathbb{C}P^1$ may be equivariantly embedded into $\mathbb{R}^3$, which is not a surprise.

Example: $M = \mathbb{C}P^2$, $P = \Delta^2$.

$$\mathcal{Z}_P = S^5 = \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3.$$

The group $K = \{(t, t, t) \in T^3\}$ acts freely on $\mathcal{Z}_P$.

Example: $M = \mathbb{C}P^2$, $P = \Delta^2$.

$$\mathcal{Z}_P = S^5 = \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3.$$

The group $K = \{(t,t,t) \in T^3\}$ acts freely on $\mathcal{Z}_P$.

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \; W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\Lambda^T W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Example: $M = \mathbb{C}P^2$, $P = \Delta^2$.

$$\mathcal{Z}_P = S^5 = \{|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3.$$

The group $K = \{(t,t,t) \in T^3\}$ acts freely on $\mathcal{Z}_P$.

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix},\ W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix},$$

$$\Lambda^T W = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

We have

$$\varphi_{\mathbb{C}P^2}(z) = (z_2\bar{z}_3,\ z_1\bar{z}_3,\ z_1\bar{z}_2).$$

Example: $M = \mathbb{C}P^2$, $P = \Delta^2$.

If $z = (z_1, z_2, z_3) \in \mathcal{Z}_P$, then for $\mathbb{C}P^2$ we may also write

$$\tilde{\varphi}_{\mathbb{C}P^2}(z) = z \cdot \bar{z} = \begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 & z_1\bar{z}_3 \\ \bar{z}_1 z_2 & |z_2|^2 & z_2\bar{z}_3 \\ \bar{z}_1 z_3 & \bar{z}_2 z_3 & |z_3|^2 \end{pmatrix}$$

The components of our map $\rho_P \times \varphi_{\mathbb{C}P^2}$ are exactly the entries of this matrix' upper-triangle part.

Example: $M = \mathbb{C}P^2$, $P = \Delta^2$.

If $z = (z_1, z_2, z_3) \in \mathcal{Z}_P$, then for $\mathbb{C}P^2$ we may also write

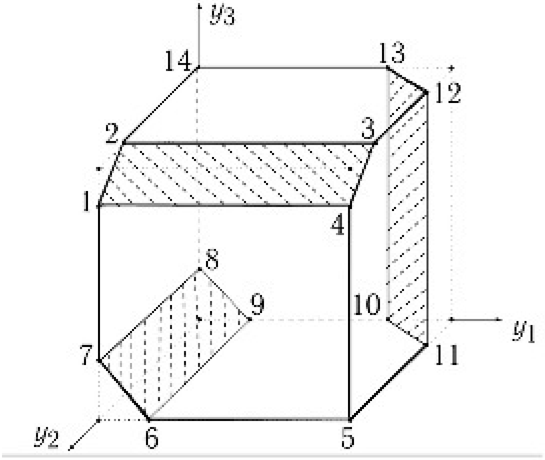
$$\tilde{\varphi}_{\mathbb{C}P^2}(z) = z \cdot \bar{z} = \begin{pmatrix} |z_1|^2 & z_1\bar{z}_2 & z_1\bar{z}_3 \\ \bar{z}_1 z_2 & |z_2|^2 & z_2\bar{z}_3 \\ \bar{z}_1 z_3 & \bar{z}_2 z_3 & |z_3|^2 \end{pmatrix}$$

The components of our map $\rho_P \times \varphi_{\mathbb{C}P^2}$ are exactly the entries of this matrix' upper-triangle part.

Since $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, the embedding space of $\mathbb{C}P^2$ is $\mathbb{R}^2 \times \mathbb{C}^3 = \mathbb{R}^8$.

Example: Stasheff polytope K_5

The vertices are enumerated according to Hamiltonian path.



Example: Stasheff polytope K_5

The matrix of 1-dimensional cones of corresponding fan is

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix}.$$

Example: Stasheff polytope K_5

Let us enumerate facets of polytope K_5 as in matrix Λ. Every edge of polytope K_5 is the intersection of exactly two facets.

Then edges of K_5 are formed by pairs (1, 2), (1, 5), (1, 6), (1, 7), (1, 8), (2, 3), (2, 6), (2, 8), (2, 9), (3, 4), (3, 5), (3, 8), (3, 9), (4, 5), (4, 6), (4, 7), (4, 9), (5, 7), (5, 8), (6, 7), (6, 9).

Example: Stasheff polytope K_5

To generate an equivariant embedding it is enough to choose the following 6 monomial maps:

$$\varphi_{12} = z_3 \bar{z}_6 \bar{z}_7 z_8$$

$$\varphi_{16} = \bar{z}_2 z_5 z_7 \bar{z}_9$$

$$\varphi_{17} = \bar{z}_2 z_3 z_5 \bar{z}_6 z_8 \bar{z}_9$$

$$\varphi_{23} = z_1 \bar{z}_4 z_8 \bar{z}_9$$

$$\varphi_{28} = \bar{z}_1 z_3 z_4 \bar{z}_6 \bar{z}_7 z_9$$

$$\varphi_{39} = z_1 z_2 \bar{z}_4 \bar{z}_5 \bar{z}_7 z_8$$

Example: Stasheff polytope K_5

Hence for Stasheff polytope K_5 we have $q = 6$,
but the total number of edges of K_5 is 21.

Example: Stasheff polytope K_5

Hence for Stasheff polytope K_5 we have $q = 6$, but the total number of edges of K_5 is 21.

The corresponding toric variety is embedded into $\mathbb{R}^3 \times \mathbb{C}^6 = \mathbb{R}^{15}$ instead of $\mathbb{R}^{45}$.

Application of 2-truncated cubes theory

The theory of 2-truncated cubes has been developed in [Buchstaber-Volodin, 2012].
An explicit construction of pairs (P, Λ) is given for quasitoric manifolds M^{2n} that are symplectic manifolds with Hamiltonian action of torus T^n where the image of moment map is the given 2-truncated cube.

Application of 2-truncated cubes theory

The theory of 2-truncated cubes has been developed in [Buchstaber-Volodin, 2012]. An explicit construction of pairs (P, Λ) is given for quasitoric manifolds M^{2n} that are symplectic manifolds with Hamiltonian action of torus T^n where the image of moment map is the given 2-truncated cube.

Well-known polytopes, such as flag nestohedra, graph-associahedra and graph-cubeahedra, are 2-truncated cubes.

Application of 2-truncated cubes theory

Problem. How can one estimate value of q in terms of sequence of 2-truncations of cube?

Signs of fixed points

Let $x \in M$ be a fixed point of torus action and $\pi(x) = v$.

Signs of fixed points

Let $x \in M$ be a fixed point of torus action and $\pi(x) = v$.

Denote by $\mathbb{C}_v \subset \mathbb{C}^q$ the space corresponding to n isotropy subgroups of edges starting in vertex $v \in R$.

Signs of fixed points

Let $x \in M$ be a fixed point of torus action and $\pi(x) = v$.

Denote by $\mathbb{C}_v \subset \mathbb{C}^q$ the space corresponding to n isotropy subgroups of edges starting in vertex $v \in R$.

Consider natural map

$$\varphi_{x,*} \colon T_x(M) \longrightarrow \mathbb{R}^n \times \mathbb{C}^q \longrightarrow \mathbb{C}_v$$

that is composition of differential and projection map.

Signs of fixed points

Denote by W_v the matrix formed by corresponding columns of W and define $S_v = W_v^T \Lambda_v$.

Signs of fixed points

Denote by W_v the matrix formed by corresponding columns of W and define $S_v = W_v^T \Lambda_v$.

Theorem 2. The sign of

$$\det(\varphi_{x,*}) \cdot \det(S_v)$$

equals to the sign of fixed point $x \in M$ as defined originally in [Buchstaber-Panov-Ray, 2007].

Signs of fixed points

We may now define signs of fixed points of M in purely geometrical terms of equivariant embedding $\rho_P \times \varphi_M$ and associated representation Φ.

Almost complex structures

Theorem (K., 2009). A quasitoric manifold M admits a T^n-invariant almost complex structure if and only if all signs of its fixed points, as defined in Theorem 2, are positive.

Almost complex structures

Theorem (K., 2009). A quasitoric manifold M admits a T^n-invariant almost complex structure if and only if all signs of its fixed points, as defined in Theorem 2, are positive.

Hence, the criteria for existence of invariant almost complex structures may be reformulated in terms of equivariant embedding constructed in Theorem 1.

Example: $\mathbb{C}P^1$

We have a natural inclusion

$$\mathbb{C}P^1 \xrightarrow{\pi \times \varphi} \mathbb{R} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C},$$

Example: $\mathbb{C}P^1$

The normal bundle $\nu(\mathbb{C}P^1)$ of

$$\mathbb{C}P^1 \subset \mathbb{R} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$$

is trivial, hence Chern class $c_1(\nu) = e(\nu)$ is zero.

Example: $\mathbb{C}P^1$

The normal bundle $\nu(\mathbb{C}P^1)$ of

$$\mathbb{C}P^1 \subset \mathbb{R} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$$

is trivial, hence Chern class $c_1(\nu) = e(\nu)$ is zero.

We see that almost complex structure on $\mathbb{C}P^1$ is not inherited from $\mathbb{C} \times \mathbb{C}$, since otherwise

$$c(\tau(\mathbb{C}P^1)) \cdot c(\nu(\mathbb{C}P^1)) = c(\mathbb{C} \times \mathbb{C}) = 1,$$

which is not true.

- Michael F. Atiyah. Convexity and commuting Hamiltonians. Bull. London Math. Soc. 14 (1982), no. 1, 1-15.
- Michael W. Davis and Tadeusz Januszkiewicz. Convex polytopes, Coxeter orbifolds and torus actions. Duke Math. J., 62 (1991), no. 2, 417-451.
- V. M. Buchstaber, T. E. Panov. Torus actions, combinatorial topology, and homological algebra. Russian Math. Surveys, 55:5 (2000), 825–921

- V. Ginzburg, V. Guillemin, Y. Karshon. Moment maps, cobordisms and Hamiltonian group actions. American Mathematical Society, 2002.
- Victor M. Buchstaber and Taras E. Panov. Torus Actions and Their Applications in Topology and Combinatorics. University Lectures Series, vol.24, American Mathematical Society, Providence, RI, 2002, 152 pages.
- Victor M. Buchstaber, Taras E. Panov and Nigel Ray. Spaces of polytopes and cobordism of quasitoric manifolds. Moscow Math. J. 7 (2007), no. 2, 219-242; arXiv:math.AT/0609346.

- Victor M Buchstaber and Nigel Ray. An Invitation to Toric Topology: Vertex Four of a Remarkable Tetrahedron. MIMS EPrint: 2008.31. http://www.manchester.ac.uk/mims/eprints
- A. Kustarev. Equivariant almost complex structures on quasitoric manifolds. (Russian) Tr. Mat. Inst. Steklova 266 (2009), Geometriya, Topologiya i Matematicheskaya Fizika. II, 140–148; translation in Proc. Steklov Inst. Math. 266 (2009), no. 1, 133-141.

V. M. Buchstaber, V. D. Volodin. Combinatorial 2-truncated cubes and applications. Associahedra, Tamari Lattices and Related Structures, Tamari Memorial Festschrift, Progress in Mathematics, 299, Birkhäuser, Basel, 2012, 161-186.

Michael W. Davis. When are two Coxeter orbifolds diffeomorphic? The Michigan Mathematical Journal 63 (2014), no. 2, 401-421.

Victor M. Buchstaber, Svjetlana Terzić. (2n, k)-manifolds and applications. Mathematisches Forschungsinstitut Oberwolfach Report No. 27/2014, p.11-14, DOI:10.4171/OWR/2014/27