On $\mathbb{A}^1$-contractibility of Koras-Russell threefolds

Amalendu Krishna

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1. Motivation And Background of the problem

In this talk, all vector spaces, varieties and manifolds will be over complex numbers.

Consider the (trivial) linear algebra problem:

Question

Let V_1, V_2 and W be three finite-dimensional vector spaces such that $V_1 \times W \simeq V_2 \times W$. Is $V_1 \simeq V_2$?

As we all know, the affine algebra (or affine algebraic geometry) is the polynomial extension of linear algebra. In other words, we study solution sets of polynomial equations rather than only the linear equations. From this perspective, the natural generalization of the above question in affine geometry would be the following.

Question

Let V_1, V_2 and W be smooth affine varieties such that $V_1 \times W \simeq V_2 \times W$. Is $V_1 \simeq V_2$?

In complex affine geometry, this question can be phrased as follows.

Question

Let V_1, V_2 and W be complex Stein manifolds such that $V_1 \times W \simeq V_2 \times W$. Is $V_1 \simeq V_2$?

We shall concentrate on the algebraic side of these questions.

The weakest possible, but still non-trivial, case of the above question is the following famous Cancellation problem posed by Zariski.

Question (Zariski's Cancellation problem)

Let V be a smooth affine variety such that $V \times \mathbb{A}^n \simeq \mathbb{A}^{d+n}$ for some $n \geq 1$. Is $V \simeq \mathbb{A}^d$?

It is not difficult to see that if the answer to this question is 'YES' (or 'NO') for $n = 1$, then it is 'YES' (or 'NO') for all $n \geq 1$. So we concentrate on $n = 1$ case.

Definition

We shall say that a variety V is stably an affine space if $V \times \mathbb{A}^n \simeq \mathbb{A}^{d+n}$ for some n.

The following is the status of the Cancellation problem.

1. $d = 1$ (solved by Abhyankar, Eakin and Heinzer in 1972).
2. $d = 2$ (solved by Fujita and independently by Miyanishi and Sugie in 1980).
3. $n \geq 3$ (open).

In early nineties, Peter Russell produced the following threefold.

$$V = \{(x, y, z, t) \in \mathbb{A}^4 | x + x^2 y + z^2 + t^3 = 0\}.$$

This threefold is now famously known as the Russell cubic in literature. Later on, together with Koras, he produced a huge class of affine threefolds which contains Russell cubic. These threefolds are now known as the Koras-Russell threefolds.

So what is a Koras-Russell threefold?

Recall that an algebraic action of $\mathbb{G}_m(\mathbb{C}) \simeq \mathbb{C}^*$ on a smooth affine variety is called is *hyperbolic* if it has a unique fixed point and the weights of the induced linear action on the tangent space at this fixed point are all non-zero and their product is negative.

Definition

A *Koras-Russell* threefold X is a smooth hypersurface in $\mathbb{A}^4$ which is

1. topologically contractible,
2. has a hyperbolic $\mathbb{C}^*$-action, and
3. the quotient $X//\mathbb{C}^*$ is isomorphic to the quotient of the linear $\mathbb{C}^*$-action on the tangent space at the fixed point.

Example

The Russell cubic $x + x^2y + z^2 + t^3 = 0$ is the first example of a Koras-Russell threefold. The $\mathbb{C}^*$-action on this is induced by the linear action:

$$\lambda \cdot (x, y, z, t) = (\lambda^6 x, \lambda^{-6} y, \lambda^3 z, \lambda^2 t).$$

Koras and Russell invented these threefolds out of their motivation to solve the Cancellation and the Linearization problems. They succeeded in solving the linearization problem in dimension three. But they could not get any headway towards solving the Cancellation problem although their hope was that the threefolds discovered by them would produce counter-examples to the Cancellation problem.

As no evidence was forthcoming in this direction since then, many authors began to believe that the Koras-Russell threefolds may not actually yield counter-examples to the Cancellation problem. Their belief was based on the expectation that the $\mathbb{C}^*$-action on such a threefold should produce non-trivial equivariant Grothendieck groups and this fact could then be used to show that the threefold is not stably isomorphic to $\mathbb{A}^4$. A program for reaching this goal was outlined in a paper by Asok. This goes as follows.

1. Show that $\mu_p \subsetneq \mathbb{C}^*$ acts on X with $X^{\mu_p} = X^{\mathbb{C}^*}$ for almost all (all but finitely many) primes p.
2. Show that if $f : X \to Y$ is a μ_p-equivariant map of smooth schemes with μ_p-action such that $X \to Y$ as well as $X^{\mu_p} \to Y^{\mu_p}$ are $\mathbb{A}^1$-weak-equivalences (in the sense of Morel-Voevodsky), then f induces an isomorphism of the μ_p-equivariant Grothendieck groups.
3. Show that the $\mathbb{C}^*$-equivariant Grothendieck group of X is non-trivial, that is, $K_0^{\mathbb{C}^*}(X) \ncong K_0^{\mathbb{C}^*}(pt)$.
4. Show that the μ_p-equivariant Grothendieck group of X is non-trivial for almost all primes p.

An important thing to notice here is that it is enough to verify the steps (2)-(4) with rational coefficients to complete Asok's program.

This will allow one to conclude that X is not $\mathbb{A}^1$-contractible and hence not stably isomorphic to $\mathbb{A}^4$. Let us see how this works.

The point is that if X is indeed $\mathbb{A}^1$-contractible, then it having a unique $\mathbb{C}^*$-fixed point will mean that there is a prime p such that $K_0^{\mu_p}(X)$ is non-trivial and the fixed-point loci of X for all subgroups of μ_p are $\mathbb{A}^1$-contractible. The step (2) will now contradict the non-triviality of $K_0^{\mu_p}(X)$. Finally, it is easy to see in such case that X can not be stably an affine space. For,

$$X \times \mathbb{A}^m \simeq \mathbb{A}^{m+3} \Rightarrow X \sim X \times \mathbb{A}^m \sim \mathbb{A}^{m+3} \sim pt.$$

It is therefore important to know whether all the steps of Asok's program as outlined above can be verified.

Our goal:

To verify all the steps of Asok's program. If we succeed, the story of Koras-Russell threefold will be over as far as the Cancellation problem is concerned. If not, then we will be back to believing that these threefolds are still in the game.

So we began checking these steps and arrived at the following conclusions.

Proposition

Let $\mathbb{C}^*$ act on affine scheme X. Then there is a finite set of positive integers S such that $X^{\mu_n} = X^{\mathbb{C}^*}$ whenever n is a positive integer prime to all elements of S. Here, $\mu_n \subsetneq \mathbb{C}^*$ is the natural inclusion of algebraic groups.

Proof.

This is easy and known. The idea is to embed X equivariantly into an affine space with linear $\mathbb{C}^*$-action. This reduces to the case of linear $\mathbb{C}^*$-action on $\mathbb{A}^n$. Since we know all such actions explicitly, the result follows. □

This shows that the step (1) of Asok's program can be achieved. The step (3) was shown by J. Bell, a student of Peter Russell. He gave a formula for $K_0^{\mathbb{C}^*}(X)$. Using this, he concluded that $K_0^{\mathbb{C}^*}(X)_{\mathbb{Q}}$ is not isomorphic to $R(\mathbb{C}^*)_{\mathbb{Q}} \simeq \mathbb{Q}[t^{\pm 1}]$. We are left with steps (2) and (4). Step (2) is the hardest part of the problem and this follows from our following stronger result.

Theorem

Let G be a finite group and let $f : X \to Y$ be a G-equivariant map of smooth schemes with G-action. Assume that for every subgroup $H \subseteq G$, the induced map on the fixed point loci $X^H \to Y^H$ is an $\mathbb{A}^1$-weak equivalence. Then the map $f^* : K_*^G(Y) \to K_*^G(X)$ is an isomorphism with rational coefficients.

Recall that for topological equivariant K-theory, the above result is true with integral coefficients and for any Lie group G. This is because it was proven by Segal that (topological) weak equivalence on the fixed point loci for all closed subgroups implies that the map f is G-homotopy equivalence. And it is not hard to prove that a G-homotopy equivalence implies isomorphism on the equivariant K-theory.

However, for algebraic varieties, a G-homotopy equivalence is, in general, not detected by the ordinary $\mathbb{A}^1$-weak equivalence on the fixed point loci for all closed subgroups of G. This fails even when G is finite. So the above theorem is much more subtle.

The main step in the proof of above theorem can be described as follows.

1. Define three equivariant topologies on the category of smooth G-varieties:
$$\tau_{en} \subsetneq \tau_{fp} \subsetneq \tau_{ee}.$$
2. It is relatively 'easier' to show that the equivariant K-theory is representable in the first homotopy category.
3. Show that τ_{en} and τ_{ee} are so related that the Mayer-Vietoris for rational equivariant K-theory in τ_{en} implies so for τ_{ee} (because these two differ only by some finite coverings). In particular, rational equivariant K-theory is representable in τ_{ee}.
4. Conclude that it is representable in τ_{fp} which is the fixed point topology.

So the conclusion so far is that the steps (1) - (3) of the above program can be achieved. We are now left with (4).
Unfortunately, this is where the program breaks down. We show the following.

Theorem

Let p be a prime number and let μ_p act on a Koras-Russell threefold X via the inclusion $\mu_p \subsetneq \mathbb{C}^*$. Let $K_0^{\mu_p}(X)$ denote the Grothendieck group of μ_p-equivariant vector bundles on X. Then the map $R(\mu_p) \to K_0^{\mu_p}(X)$ is an isomorphism for all but a finite set of primes p.

For primes away from those above, we give an explicit expression as well.

It follows from a result of Bass and Haboush that every exact sequence of μ_p-equivariant vector bundles on X splits as a direct sum of μ_p-equivariant sheaves. Combining this with Theorem 0.10, we get:

Corollary

Let p be a prime number and let μ_p act on a Koras-Russell threefold X via the inclusion $\mu_p \subsetneq \mathbb{C}^*$. Then for all but a finite set of primes, every μ_p-equivariant vector bundle on X is stably trivial. That is, given any μ_p-equivariant vector bundle E on X, there are μ_p-representations F_1 and F_2 such that $E \oplus F_1 \simeq F_2$.

How to prove this theorem?
The proof is obtained using the following main steps.

1. Show that the restriction map $K_0^{\mathbb{C}^*}(X) \to K_0^{\mu_p}(X)$ induces an isomorphism
$$K_0^{\mathbb{C}^*}(X) \underset{R(\mathbb{C}^*)}{\otimes} R(\mu_p) \xrightarrow{\simeq} K_0^{\mu_p}(X).$$
2. Combine this result, Bell's computations and use some algebraic argument to conclude the theorem. This algebraic argument involves understanding the nature of various cyclotomic polynomials and their relations, especially the nature of an ideal generated by two or more cyclotomic polynomials in the polynomial ring $\mathbb{Z}[t]$.

Conclusion: The Koras-Russell threefolds might still produce counter-examples to the Cancellation problem!
To verify this, one would first like to know whether they are $\mathbb{A}^1$-contractible. To strengthen this view further, we show the following stable contractibility result.

Let $T = (\mathbb{CP}^1, \infty)$ denote the pointed complex projective line. Let $(X, 0)$ denote a Koras-Russell threefold, pointed by the unique fixed point 0. For $n \geq 0$, let $\Sigma^n_T(X, 0)$ denote the n-th T-suspension of X.

Theorem

Let X be a Koras-Russell threefold. Then there exists an integer $n \geq 0$ such that $\Sigma^n_T(X, 0)$ is $\mathbb{A}^1$-contractible.

This yields a strong evidence that a Koras-Russell threefold may indeed be $\mathbb{A}^1$-contractible.

This theorem also provides the first example of a smooth affine variety which is not isomorphic to an affine space but whose finite T-suspension is $\mathbb{A}^1$-contractible.

Observe that in topology, this is not a serious assertion. However, as is well known, $\mathbb{A}^1$-contractibility of an algebraic variety is supposedly a much stronger assertion. For example, one can prove that a smooth affine curve which is $\mathbb{A}^1$-contractible, must be isomorphic to $\mathbb{A}^1$. (Such a thing clearly fails for topological spaces without further structure.)

Proof of this theorem:

This theorem is proven in several steps which can be described as follows. The most crucial step is to show the following theorem.

Theorem

Let X be a Koras-Russell threefold and let Y be any smooth affine variety. Then the map $H^{*,*}(Y) \to H^{*,*}(X \times Y)$ of motivic cohomology is an isomorphism. In particular, the motivic cohomology of X is trivial.

This is proved by realizing the threefold X as a flat family over $\mathbb{A}^1$ which consists of a degeneration of the algebraic affine plane into a topological affine plane. Once we have this, we use various properties of the motivic cohomology like the homotopy invariance and localization to prove the above theorem.

Example

Suppose X is the Russell cubic given by $x + x^2y + z^2 + t^3 = 0$. We can then define a map $\pi : X \to \mathbb{C}$ by $\pi(x, y, z, t) = x$. One can then see that $\pi^{-1}(\mathbb{C}^*) \simeq \mathbb{C}^* \times \mathbb{C}^2$ and $\pi^{-1}(0) \simeq \mathbb{C} \times \mathrm{Cusp}$, which is topologically $\mathbb{C}^2$.

Already as a consequence of this result, one gets the following conclusion.

Corollary

Let X be a threefold as in the previous theorem. Then every algebraic vector bundle on X is trivial.

This question for Koras-Russell threefolds was posed by Koras and Russell (1997) and was answered for a smaller class of threefolds by Murthy (2001).
Once we have the relative vanishing result for the motivic cohomology, we invoke a hard result of Morel from which it follows that if a smooth affine variety X satisfies the condition of the previous theorem, then $\Sigma^{\infty}_{T}(X, 0)$ is the trivial object of the stable homotopy category.

We then use certain techniques of stable homotopy theory and some more argument to conclude that $\Sigma^n_T(X, 0)$ is unstably contractible for some $n \geq 0$.

Work in Progress: In this work, we wish to complete our program of showing that X is $\mathbb{A}^1$-contractible. The steps are as follows.

1. Show that X is $\mathbb{A}^1$-connected and $\mathbb{A}^1$-simply connected.
2. Show that X is homologically trivial. This requires one to know that X is cohomologically trivial, which we have shown above.
3. Use Whitehead theorem to conclude that X has no non-trivial higher homotopy groups and hence is contractible.

Some further questions:

(1) We have seen that a Koras-Russell threefold has no non-trivial μ_p-equivariant K-theory. One reason these threefolds are so complicated to deduce any geometric information is that they have a torus action which is not maximal. So the question is: can we realize this $\mathbb{C}^*$-action as a restriction of a larger torus action?

If this is true, we can then find many more finite subgroups of the larger torus for which the equivariant K-theory may be non-trivial. That will allow one conclude important information about the threefold.

(2) The second question is that we have seen that a Koras-Russell threefold has no non-trivial vector bundle and every μ_p-equivariant vector bundle is stably trivial. One can ask: Is every μ_p-equivariant vector bundle on X actually trivial?

Notice that if X was $\mathbb{A}^3$, then the solution of the linearization problem by Koras and Russell and a result of Masuda, Moser-Jauslin and Petrie would say that all μ_p-equivariant vector bundles are indeed trivial.

THANK YOU DAEJEON