Geometric T^k-equivariant complex bordism

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Tangentially stably complex T^k-manifolds

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A *tangentially stably complex T^k-manifold M* is a smooth compact T^k-manifold with a complex T^k-structure on

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for some N, where the trivial bundle $\mathbb{R}^N$ has trivial T^k-action.

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When $N = 0$ we have a tangentially almost complex T^k-structure.

Proposition

Suppose M is a tangentially stably complex T^k-manifold. Then each component of the fixed point set M^{T^k} is a tangentially stably complex T^k-manifold and its normal bundle in M is a complex T^k-bundle.

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Proof. Observe that $\tau(M^{T^k}) = (\tau(M)|_{M^{T^k}})^{T^k}$ as real vector bundles. Then

$$\zeta := (\tau(M)\oplus\mathbb{R}^N)|_{M^{T^k}} = \tau(M)|_{M^{T^k}}\oplus\mathbb{R}^N = \tau(M^{T^k})\oplus\nu(M^{T^k}, M)\oplus\mathbb{R}^N.$$

Since $\zeta^{T^k} = \tau(M^{T^k}) \oplus \mathbb{R}^N$, we have that $(\zeta^{T^k})^\perp = \nu(M^{T^k}, M)$ is a complex T^k-bundle.

Suppose we have an isolated fixed point $p \in M^{T^k}$. We have a complex T^k-structure on

$$(\tau(M) \oplus \mathbb{R}^N)|_p \cong T_pM \oplus \mathbb{R}^N \cong \tau(p) \oplus \nu_p^M \oplus \mathbb{R}^N.$$

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$$T_pM = V_1(p) \oplus \cdots \oplus V_n(p), \quad \text{where } V_i(p) \in \mathrm{Hom}(T^k, S^1) \cong \mathbb{Z}^k.$$

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Notice T_pM has two orientations:

- one from its complex structure
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Definition (Sign of p)

For each isolated fixed point p, the *sign of p* is given by

$$\sigma(p) := \begin{cases} +1, & \text{if the two orientations coincide;} \\ -1, & \text{if the two orientations differ.} \end{cases}$$

GKM condition

We will assume that:

1. the T^k-action is effective, which implies that $k \leq n$;
2. M^{2n} has only isolated fixed points;
3. $V_1(p), \ldots, V_n(p)$ are pairwise linearly independent, for all $p \in M^{T^k}$.

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Therefore, the *one-skeleton*

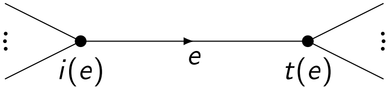
$$M_1 := \{x \in M \mid \dim T^k_x \geq n - 1\}$$

consists only of T^k-invariant embedded 2-spheres, each of which contains exactly two fixed points. It has the combinatorial structure of a graph.

Generalised GKM graphs

Let

- Γ be an n-valent connected graph with $n \geq 1$.
- $\mathcal{V}(\Gamma)$ denote the set of vertices.
- $\mathcal{E}(\Gamma)$ denote the set of *oriented* edges.



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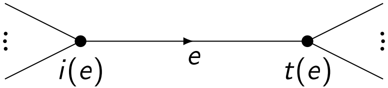

For $p \in \mathcal{V}(\Gamma)$, define

$$\mathcal{E}(\Gamma)_p := \{e \in \mathcal{E}(\Gamma) \mid i(e) = p\}.$$

Definition (Axial Function)

An *axial function* is a map

$$\alpha\colon \mathcal{E}(\Gamma) \longrightarrow \mathrm{Hom}(T^k, S^1) \cong \mathbb{Z}^k,$$

satisfying the following conditions:

1. $\alpha(\bar{e}) = \pm\alpha(e)$;
2. elements of $\alpha(\mathcal{E}(\Gamma)_p)$ are pairwise linearly independent;
3. $\alpha(\mathcal{E}(\Gamma)_{t(e)}) \equiv \alpha(\mathcal{E}(\Gamma)_{i(e)}) \bmod \alpha(e)$, for any $e \in \mathcal{E}(\Gamma)$.

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We also require an *orientation*, which is a map

$$\sigma\colon \mathcal{V}(\Gamma) \longrightarrow \{\pm 1\},$$

satisfying

$$\sigma(i(e)) = \sigma(t(e)) \iff \alpha(e) = -\alpha(\bar{e}).$$

Example (Tangentially stably complex T^k-manifold)

Let M^{2n} be one of our manifolds. Define an n-valent graph Γ_M where

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- $\mathcal{E}(\Gamma_M) = \{$2-dim submanifolds of M fixed by a $T^{k-1} \leq T^k\}$.

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Every $S \in \mathcal{E}(\Gamma_M)$ is diffeomorphic to a sphere and contains exactly two T^k-fixed points. The summands $T_pM = V_1(p) \oplus \cdots \oplus V_n(p)$ correspond to the edges $\mathcal{E}(\Gamma_M)_p$. We assign each $e \in \mathcal{E}(\Gamma_M)_p$ to its corresponding $V_i(p)$.

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$$\alpha_M \colon \mathcal{E}(\Gamma_M) \longrightarrow \mathrm{Hom}(T^k, S^1),$$

which satisfies the three conditions of being an axial function and we get a generalised GKM graph (Γ_M, α_M).

Example (Orientation)

Set $\sigma_M(p)$, for $p \in M^{T^k} = \mathcal{V}(\Gamma_M)$, to agree with the definition of the sign of an isolated fixed point of a tangentially stably complex T^k-manifold. So we get an oriented generalised GKM graph $(\Gamma_M, \alpha_M, \sigma_M)$.

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Example (T^2-action on S^6)

There is a non-integrable almost complex structure on

$$S^6 \cong G_2/SU(3)$$

and we get a smooth action of the common maximal torus T^2 with two isolated fixed points.

Equivalence relation

When the manifold is nice we can calculate Betti numbers, $H_G^*(M)$ and $K_G^*(M)$ from the labelled graph.

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When the manifold is nice we can calculate Betti numbers, $H^*_G(M)$ and $K^*_G(M)$ from the labelled graph.
For an oriented generalised GKM graph (Γ, α, σ) we define a polynomial:

$$f(\Gamma, \alpha, \sigma) := \sum_{p \in \mathcal{V}(\Gamma)} \sigma(p) \prod_{i=1}^{n} V_i(p) \in \mathbb{Z}[J_k],$$

where J_k is the set of non-trivial irreducible T^k-representations.

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We define an equivalence relation on the set $G_k(n)$ of oriented generalised GKM graphs:

$$(\Gamma_1, \alpha_1, \sigma_1) \sim (\Gamma_2, \alpha_2, \sigma_2) \Leftrightarrow f(\Gamma_1, \alpha_1, \sigma_1) = f(\Gamma_2, \alpha_2, \sigma_2).$$

Connected sum of graphs

We can define the *connected sum* $\Gamma_1 \# \Gamma_2$ of two graphs Γ_1 and Γ_2 at vertices

$$p_1 \in \Gamma_1 \quad \text{and} \quad p_2 \in \Gamma_2$$

when

$$\alpha(\mathcal{E}_{p_1}) = \alpha(\mathcal{E}_{p_2}) \quad \text{and} \quad \sigma(p_1) = -\sigma(p_2)$$

by deleting the two vertices and joining up the matching edges.

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by deleting the two vertices and joining up the matching edges. We then have that

$$f(\Gamma_1 \# \Gamma_2) = f(\Gamma_1) + f(\Gamma_2).$$

Equivariant complex cordism

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Let $\mathcal{F}_*^{T^k}$ denote the bordism ring of pairs (X, ν), where X is a stably complex manifold and ν is a complex T^k-bundle over X without trivial irreducible factors in the fibres.

By restricting to normal data around the fixed point set we obtain a **monomorphism**

$$\varphi\colon \Omega_*^{U:T^k} \longrightarrow \mathcal{F}_*^{T^k}$$

given by

$$\begin{aligned}\varphi[M] &= [M^{T^k}, \nu(M^{T^k}, M)] \\ &= \sum_{\substack{F \subset M^{T^k} \\ F \text{ connected}}} [F, \nu(F, M)].\end{aligned}$$

Definition

Let $\mathcal{Z}_{2n}^{U:T^k}$ denote the subgroup of $\Omega_{2n}^{U:T^k}$ given by elements that can be represented by a tangentially stably complex T^k-manifold with only isolated fixed points where the T^k-action is effective and satisfies the GKM condition.

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Our monomorphism can now be written as

$$\varphi\colon \mathcal{Z}_*^{U:T^k} \longrightarrow \mathbb{Z}[J_k]$$

$$[M] \longmapsto \sum_{p \in M^{T^k}} \sigma(p) \prod_{i=1}^{n} V_i(p),$$

where $T_pM = V_1(p) \oplus \cdots \oplus V_n(p)$.

The universal toric genus

There is a multiplicative monomorphism

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Then by the work of Quillen there exists a Gysin homomorphism that gives us a map

$$(1 \times_{T^k} \pi_M)_* \colon MU^*(ET^k \times_{T^k} M) \longrightarrow MU^{*-2n}(BT^k),$$

and we define the universal toric genus as

$$\Phi[M^{2n}] := (1 \times_{T^k} \pi_M)_* 1 \in MU^{-2n}(BT^k).$$

Let $S \subset MU^*(BT^k)$ be the multiplicative set generated by Euler classes $e(V) \in MU^2(BT^k)$ of bundles

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$$\begin{array}{ccc} \Omega_*^{U:T^k} & \xrightarrow{\Phi} & MU^*(BT^k) \\ \varphi \downarrow & & \downarrow \\ \mathcal{F}_*^{T^k} & \xrightarrow{S^{-1}\Phi} & S^{-1}MU^*(BT^k) \end{array}$$

where $S^{-1}MU^*(BT^k)$ is the localisation.

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which is a pullback square. This then gives us the following fixed point formula:

$$\Phi[M^{2n}] = \sum_{p \in M^{T^k}} \sigma(p) \prod_{i=1}^{n} \frac{1}{e(V_i(p))} \in MU^*(BT^k).$$

We can now extend this square using a Boardman homomorphism. That is, we have a nice natural transformation of multiplicative cohomology theories

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Theorem (Hattori)

There is a commutative pullback square with all maps injective

$$\begin{array}{ccc} \mathcal{Z}_*^{U:T^k} & \xrightarrow{\quad\Psi\quad} & K^*(BT^k)[\![\mathbf{t}]\!] \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\lambda} \\ \mathbb{Z}[J_k] & \xrightarrow{\quad S^{-1}\Psi\quad} & S^{-1}K^*(BT^k)[\![\mathbf{t}]\!] \end{array}$$

where $\mathbf{t} = (t_1, t_2, \dots)$ *is a sequence of indeterminates.*

The coefficients of $\Psi[M]$ are known as *equivariant K-theory characteristic numbers* for M.

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Theorem (D.)

Every polynomial f coming from an oriented generalised GKM graph satisfies

$$(S^{-1}\Psi)(f) \in \lambda(K^*(BT^k)[\![\mathbf{t}]\!]).$$

When $n = k$

We are now in the setting of torus manifolds. That is, we only need to assume that the T^n action is effective: the GKM condition is automatically satisfied as

$$T_pM = V_1(p) \oplus \cdots \oplus V_n(p) \quad \text{forms a basis of } \mathbb{Z}^n.$$

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Consider the free exterior $\mathbb{Z}$-algebra on the set J_n:

$$\Lambda(J_n),$$

e.g. $V \wedge V = 0 \quad \text{and} \quad V \wedge W = -W \wedge V.$

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Definition (Faithful Polynomials)

We call an exterior polynomial in $\Lambda^n(J_n)$ *faithful* if the indeterminates from each monomial form a basis of $\mathbb{Z}^n$.

Torus Polynomials

Suppose (Γ, α, σ) is an oriented torus graph. For a vertex p, order the basis elements $\alpha(\mathcal{E}(\Gamma)_p) = \{\alpha(e_1), \ldots, \alpha(e_n)\}$ so that

$$\det[\alpha(e_1) \cdots \alpha(e_n)] = \sigma(p).$$

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This defines a faithful exterior monomial

$$\mu_p = \alpha(e_1) \wedge \cdots \wedge \alpha(e_n) \in \Lambda^n(J_n), \quad \forall p \in \mathcal{V}(\Gamma).$$

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Definition (Torus Polynomial)

The *torus polynomial* of an oriented torus graph (Γ, α, σ) is the faithful exterior polynomial

$$g(\Gamma, \alpha, \sigma) := \sum_{p \in \mathcal{V}(\Gamma)} \mu_p \in \Lambda^n(J_n).$$

Definition

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For each faithful exterior polynomial $h \in \Lambda^n(J_n)$ we can obtain a *dual polynomial* $h^* \in \Lambda^n(J_n^*)$.

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We now define a chain complex $(\Lambda^k(J_n^*), d_k)$ as follows: for each monomial $s_1 \wedge \cdots \wedge s_k \in \Lambda^k(J_n^*)$, with all $s_i \in J_n^*$,

$$d_k(s_1 \wedge \cdots \wedge s_k) := \begin{cases} \sum_{i=1}^{k} (-1)^{i+1} s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_k, & \text{if } k > 1; \\ 1 & \text{if } k = 1. \end{cases}$$

and $d_0(1) = 0$. It is easy to see that $d^2 = 0$.

Theorem

Theorem (D.)

Let $h \in \Lambda^n(J_n)$ *be a faithful polynomial. Then* $h = g(\Gamma, \alpha, \sigma)$ *is the torus polynomial of an oriented torus graph if and only if* $d(h^*) = 0$.

Theorem

Theorem (D.)

Let $h \in \Lambda^n(J_n)$ be a faithful polynomial. Then $h = g(\Gamma, \alpha, \sigma)$ is the torus polynomial of an oriented torus graph if and only if $d(h^*) = 0$.

Let K_n denote the abelian group of all faithful exterior polynomials $h \in \Lambda^n(J_n)$ such that $d(h^*) = 0$.

We obtain the commutative diagram of abelian groups

$$\begin{array}{ccc} & \mathcal{Z}_{2n}^{U:T^n} & \\ {}^{g}\swarrow & & \searrow^{\varphi} \\ K_n & \xrightarrow{\quad h \quad} & \mathbb{Z}[J_n] \end{array}$$

where $h(s_1 \wedge \cdots \wedge s_n) = \det[s_1 \cdots s_n] s_1 \cdots s_n$ for a faithful monomial $s_1 \wedge \cdots \wedge s_n \in \Lambda^n(J_n)$.

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Corollary

We have an isomorphism of abelian groups

$$\mathcal{Z}_{2n}^{U:T^n} \cong K_n.$$

Define the graded rings

$$\Xi_* := \bigoplus_{n \geq 0} \mathcal{Z}_{2n}^{U:T^n} \qquad \cong \qquad K_* := \bigoplus_{n \geq 0} K_n.$$

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Warning

These are non-commutative rings.

Suppose M^{2n} is a non-bounding stably complex torus manifold. Then $g[M] \in K_n$ is a non-zero faithful polynomial in $\Lambda^n(J_n)$ such that $d(g[M]^*) = 0$. Any such exterior polynomial must have at least $n+1$ monomials.

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Corollary

As a strict lower bound, $n + 1$ is the minimum number of fixed points of a non-bounding stably complex torus manifold.

Definition

A *quasitoric manifold* is an even-dimensional smooth closed manifold M^{2n} with a locally standard smooth T^n-action such that the orbit space is a simple polytope P.

Definition

A *quasitoric pair* (P, λ) consists of a combinatorial oriented simple n-polytope P and a map

$$\lambda \colon \mathcal{F}(P) \longrightarrow \mathrm{Hom}(S^1, T^n) \cong \mathbb{Z}^n$$

that satisfies:

$$\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\} \text{ forms a basis of } \mathrm{Hom}(S^1, T^n) \text{ whenever } F_{i_1} \cap \dots \cap F_{i_n} \text{ is a vertex of } P. \qquad (\star)$$

Quasitoric Pairs

We have a bijection

$$\{\text{Quasitoric manifolds with a stably complex } T^n\text{-structure}\}$$

$$\updownarrow$$

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We define a product on quasitoric pairs

$$(P_1, \lambda_1) \times (P_2, \lambda_2) := (P_1 \times P_2, \lambda_1 \times \lambda_2),$$

where the characteristic map is defined as

$$(\lambda_1 \times \lambda_2)(F_i \times P_2) = (\lambda_1(F_i), 0, \dots, 0) \quad \text{and}$$
$$(\lambda_1 \times \lambda_2)(P_1 \times F_i') = (0, \dots, 0, \lambda_2(F_i')).$$

Ring of Quasitoric Pairs

Definition (Ring of Quasitoric Pairs)

Denote the free abelian group generated by all quasitoric pairs by $\mathcal{Q}_*$, where we may interpret $+$ as disjoint union and grade $\mathcal{Q}_*$ by the dimension of the polytope.

The multiplication depends on the ordering of $P_1 \times P_2$ so $\mathcal{Q}_*$ forms a graded non-commutative ring.

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The multiplication depends on the ordering of $P_1 \times P_2$ so $\mathcal{Q}_*$ forms a graded non-commutative ring.We have a homomorphism of non-commutative graded rings

$$\mathcal{M} \colon \mathcal{Q}_* \longrightarrow \Xi_*,$$

by constructing the omnioriented quasitoric manifold associated to a quasitoric pair.

Conjecture

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The homomorphism $\mathcal{M}\colon \mathcal{Q}_* \longrightarrow \Xi_*$ is surjective, that is, every class is Ξ_* contains an omnioriented quasitoric manifold.

True for $n = 1, 2$.

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Remark

The corresponding conjecture for small covers and unoriented $(\mathbb{Z}/2)^n$-equivariant bordism has been proved by Zhi Lü and Qiangbo Tan.

Thank you!