# $\chi$-boundedness of graphs with cuts of small rank

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# Cliques and colorings

- $\chi(G) \geq \omega(G)$
- triangle-free graphs have unbounded chromatic number

# Examples

- $G$ perfect $\Rightarrow \chi(G) = \omega(G)$
  - chordal graphs, comparability graphs, linegraphs of bipartite graphs, ...
- $G$ circle-arc $\Rightarrow \chi(G) \le 2\omega(G)$
- $G$ circle $\Rightarrow \chi(G) \le 2^{\omega(G)+7}$ (Kostochka and Kratochvíl)

# $\chi$-boundedness

**Definition**

A class of graphs $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that $G \in \mathcal{G} \Rightarrow \chi(G) \leq f(\omega(G))$.

# Questions

Which hereditary graph classes are $\chi$-bounded?

**Conjecture (Gyárfás, Sumner)**

*For any tree $T$, the class of graphs that do not contain $T$ as an induced subgraph is $\chi$-bounded.*

Holds if $T$ is

- a star or a path
- a tree of radius at most two (Kierstead and Penrice)
- a subdivision of a star (Scott)

Forbidding a graph containing a cycle does not cause $\chi$-boundedness.

# Questions

**Conjecture (Scott)**

*For any $H$, the class of graphs that do not contain any subdivision of $H$ as an induced subgraph is $\chi$-bounded.*

Holds if $H$ is

- a tree (Scott)
- a cycle of length at most 5 (Scott)

**Conjecture (Gyárfás)**

*For every $k$, the class of graphs that do not contain an induced cycle of length at least $k$ is $\chi$-bounded.*

# Questions

**Conjecture (Geelen)**

*For any $H$, the class of graphs that do not contain $H$ as a vertex-minor is $\chi$-bounded.*

Equivalently,

**Conjecture**

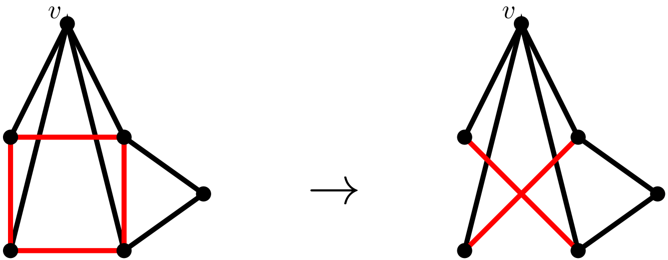
*Any proper class of graphs closed on vertex minors is $\chi$-bounded.*

# Vertex-minors

## Definition

A vertex-minor of $G$ is a graph obtained from $G$ by a sequence of vertex removals and local complementations.

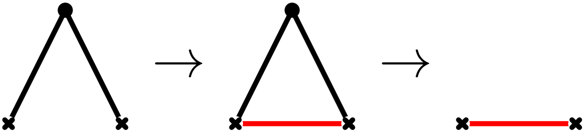
Local complementation at $v$:



## Definition

A vertex-minor of $G$ is a graph obtained from $G$ by a sequence of vertex removals and local complementations.

Induced subdivision $\Rightarrow$ vertex minor:



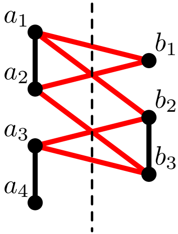
# Classes closed on vertex-minors

- circle graphs
- graphs with rank-width bounded by a constant

# Rank of an edge-cut

## Definition

Rank of an edge-cut is the rank of its adjacency matrix over $Z_2$.



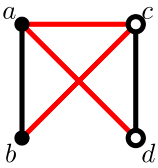

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \text{rank} = 2$$

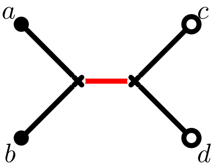
# Rank-width

- <u>rank-decomposition</u> of $G$ ... ternary tree $T$ whose leaves correspond to $V(G)$
- each edge of $T$ corresponds to an edge-cut
- <u>width</u> of $T$ ... maximum rank of the edge-cuts corresponding to $E(T)$

## Definition

<u>Rank-width</u> of a graph is the minimum width of its rank-decomposition.



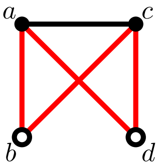

rank-width $\leq 2$

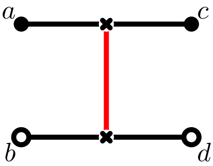
# Rank-width

- rank-decomposition of $G$ . . . ternary tree $T$ whose leaves correspond to $V(G)$
- each edge of $T$ corresponds to an edge-cut
- width of $T$ . . . maximum rank of the edge-cuts corresponding to $E(T)$

**Definition**

Rank-width of a graph is the minimum width of its rank-decomposition.




rank-width $= 1$

# Properties of rank-width

- if $H$ is a vertex-minor of $G$, then $\mathrm{rw}(H) \leq \mathrm{rw}(G)$
- $\mathrm{rw}(G) \leq \mathrm{tw}(G) + 1$
- related to clique-width: $\mathrm{rw}(G) \leq \mathrm{cw}(G) < 2^{\mathrm{rw}(G)+1}$

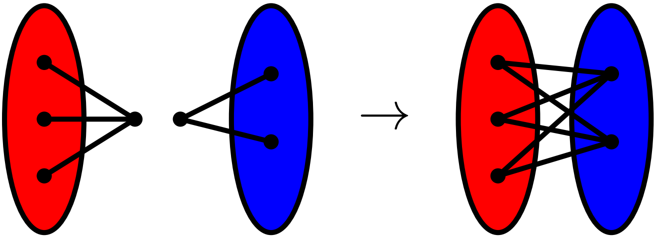
# Our results

## Theorem

*For any $k \geq 0$, the class of graphs with rank-width at most $k$ is $\chi$-bounded.*

## Theorem

*If $\mathcal{G}$ is hereditary and $\chi$-bounded, then the class of graphs obtained from graphs in $\mathcal{G}$ by a sequence of 1-joins is $\chi$-bounded.*



# Application

## Theorem (Geelen)

*Every connected graph without a vertex-minor isomorphic to the wheel $W_5$ can be obtained by 1-joins from circle graphs and graphs with at most 8 vertices.*

## Corollary

*The class of graphs without a vertex-minor isomorphic to the wheel $W_5$ is $\chi$-bounded.*

# Main result

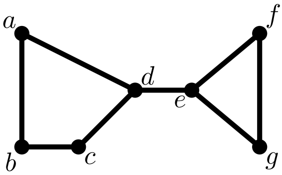
**Theorem**

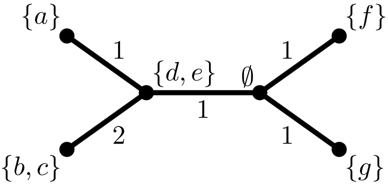
*For any hereditary $\chi$-bounded class $\mathcal{G}$ and any k $\geq$ 0, the class of graphs that have a decomposition of rank at most k over $\mathcal{G}$ is $\chi$-bounded.*

Decomposition of $G$:

- a tree $T$
- $\tau$ assigns to each $v \in V(T)$ a subset of vertices of $G$
  - $\tau(u) \cap \tau(v) = \emptyset$ if $u \neq v$
  - $\bigcup_{v \in V(T)} \tau(v) = V(G)$

Rank of the decomposition: the maximum rank of an edge-cut of $G$ corresponding to an edge of $T$.

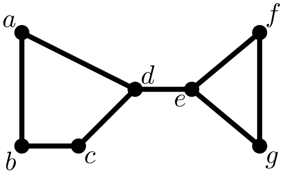


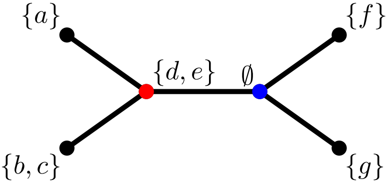



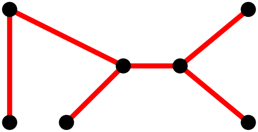
# Decomposition over class

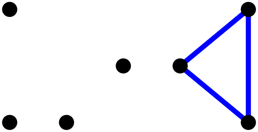
Given a decomposition $T$ of $G$:

- let $G_v$ be the graph obtained from $G$ by removing all edges whose ends lie in the same component of $T - v$
- the decomposition is <u>over $\mathcal{G}$</u> if $G_v \in \mathcal{G}$ for all $v \in V(T)$





# Related result

$G$ is a $k$-merge of $G_1$ and $G_2$ if $G_1$ and $G_2$ contain isomorphic induced subgraphs $H_1$ and $H_2$ with at most $k$ vertices and $G$ is obtained from the disjoint union of $G_1$ and $G_2$ by identifying the corresponding vertices of $H_1$ and $H_2$.

**Theorem (Chudnovsky et al.)**

*If $G$ is obtained from graphs $G_1, \ldots, G_n$ by a sequence of $k$-merges, then $\chi(G) \leq \max(\chi(G_1), \ldots, \chi(G_n)) + 2k^2 - 1$.*

**Corollary (Chudnovsky et al.)**

*If $\mathcal{G}$ is a $\chi$-bounded class of graphs and $k \geq 0$, then the class of graphs obtained from graphs in $\mathcal{G}$ by a sequence of $k$-merges is $\chi$-bounded.*

**Problem**

*Does some common generalization of the results hold?*

# Thank you for your attention

Questions?