Well-quasi-ordering, clique-width and factorial properties of graphs

Vadim Lozin

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Mathematics Institute

University of Warwick

Is there anybody going to listen to my story?

The Beatles, “Girl”

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The Beatles, “Girl”

Definition. *A class of graphs is hereditary if it is closed under taking induced subgraphs*

Observation. *A class of graphs is hereditary if and only of it can be characterized in terms of forbidden induced subgraphs.*

Bipartite graphs=Free(C_3,C_5,C_7,…) : graphs partitionable into two independent sets

Split graphs=(C_4,C_5,$2K_2$) : graphs partitionable into a clique and an independent set

The speed of hereditary properties of graphs

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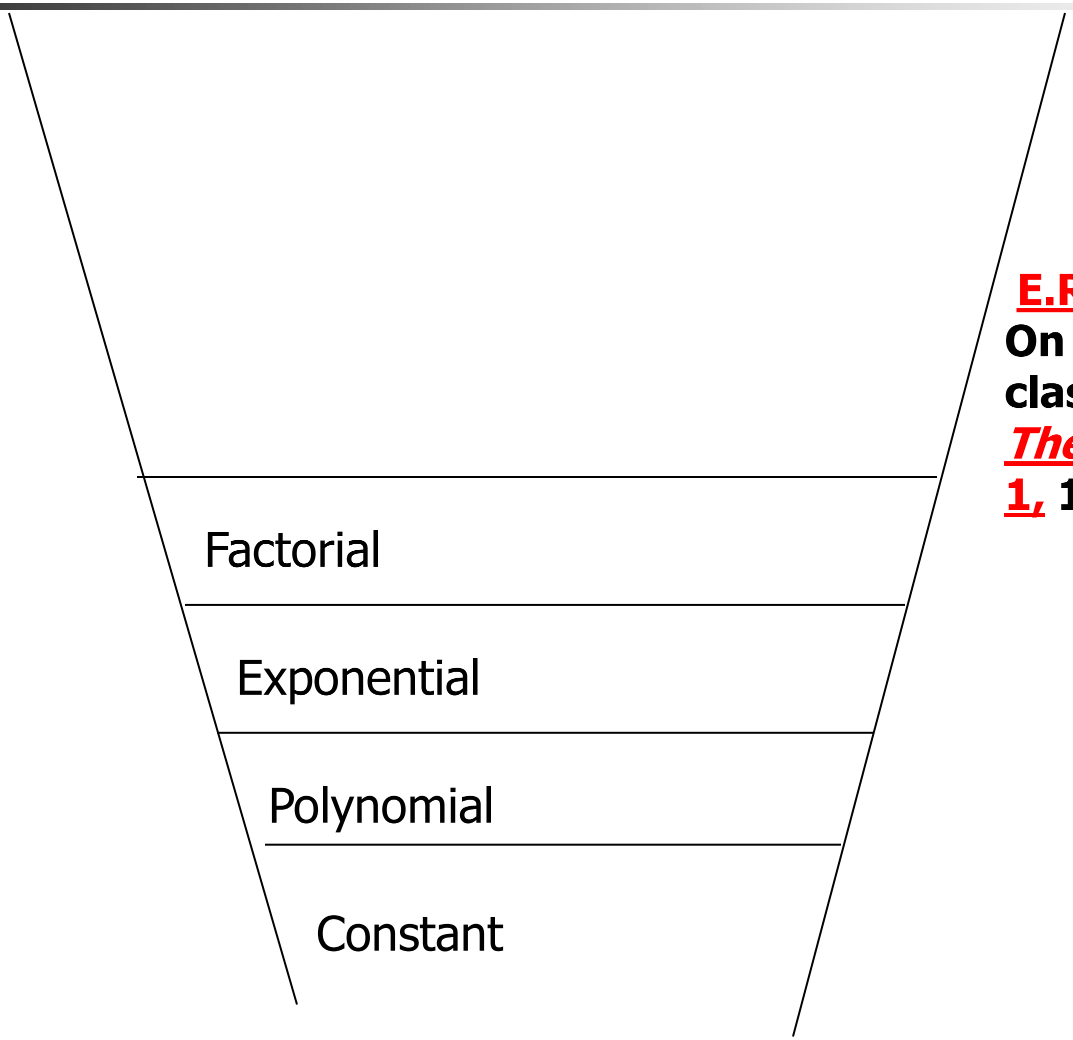
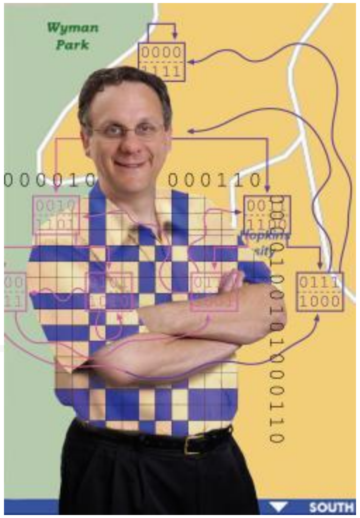
(the number of n-vertex labeled graphs)

The speed of hereditary properties of graphs.





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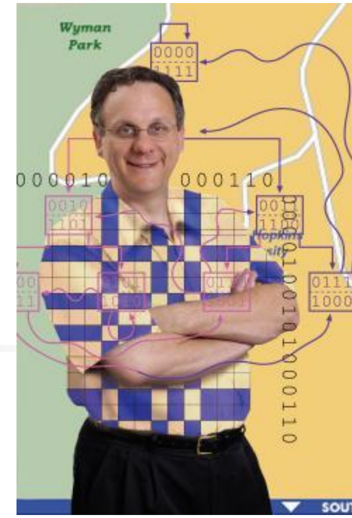
The speed of hereditary properties of graphs.

Factorial

Exponential

Polynomial

Constant



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J. Balogh, B. Bollobás, D. Weinreich,

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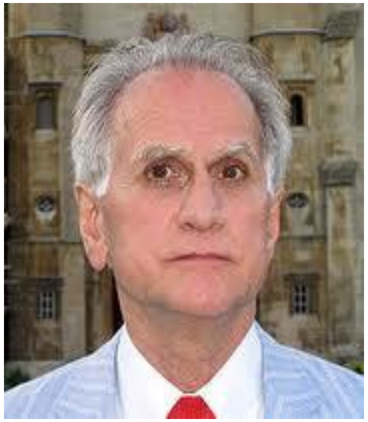
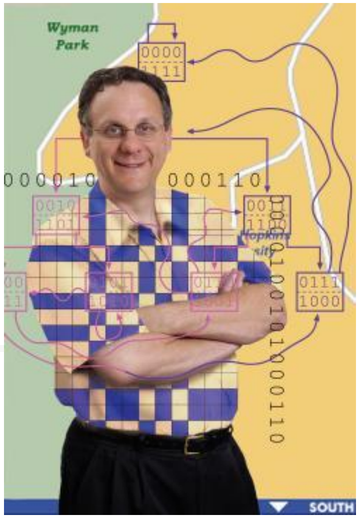
J. Combin. Theory Ser. B 79 (2000) 131–156.





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Factorial classes of graphs

- interval graphs
- permutation graphs
- line graphs
- threshold graphs
- cographs
- forests
- planar graphs
- all proper minor-closed graph classes
- all classes of graphs of bounded vertex degree

Conjecture

A class X of graphs is factorial if and only if the fastest of the following three classes is factorial:

- the class of bipartite graphs in X
- the class of co-bipartite graphs in X
- the class of split graphs in X

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Graph classes defined by forbidden induced subgraphs with at most 4 vertices

V. Lozin, C. Mayhill, V. Zamaraev **A note on the speed of hereditary properties of graphs.** *Electronic J. Combinatorics, 18 (1) 2011, #P157*.

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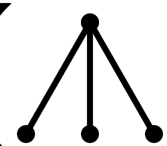
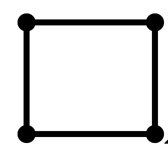
( , )-

free graphs

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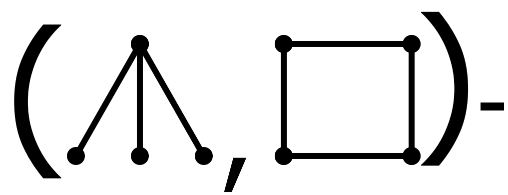
Chudnovsky, Maria; Seymour, Paul **The structure of claw-free graphs.** *Surveys in combinatorics 2005,* 153–171, London Math. Soc. Lecture Note Ser., 327, *Cambridge Univ. Press, Cambridge,* 2005.



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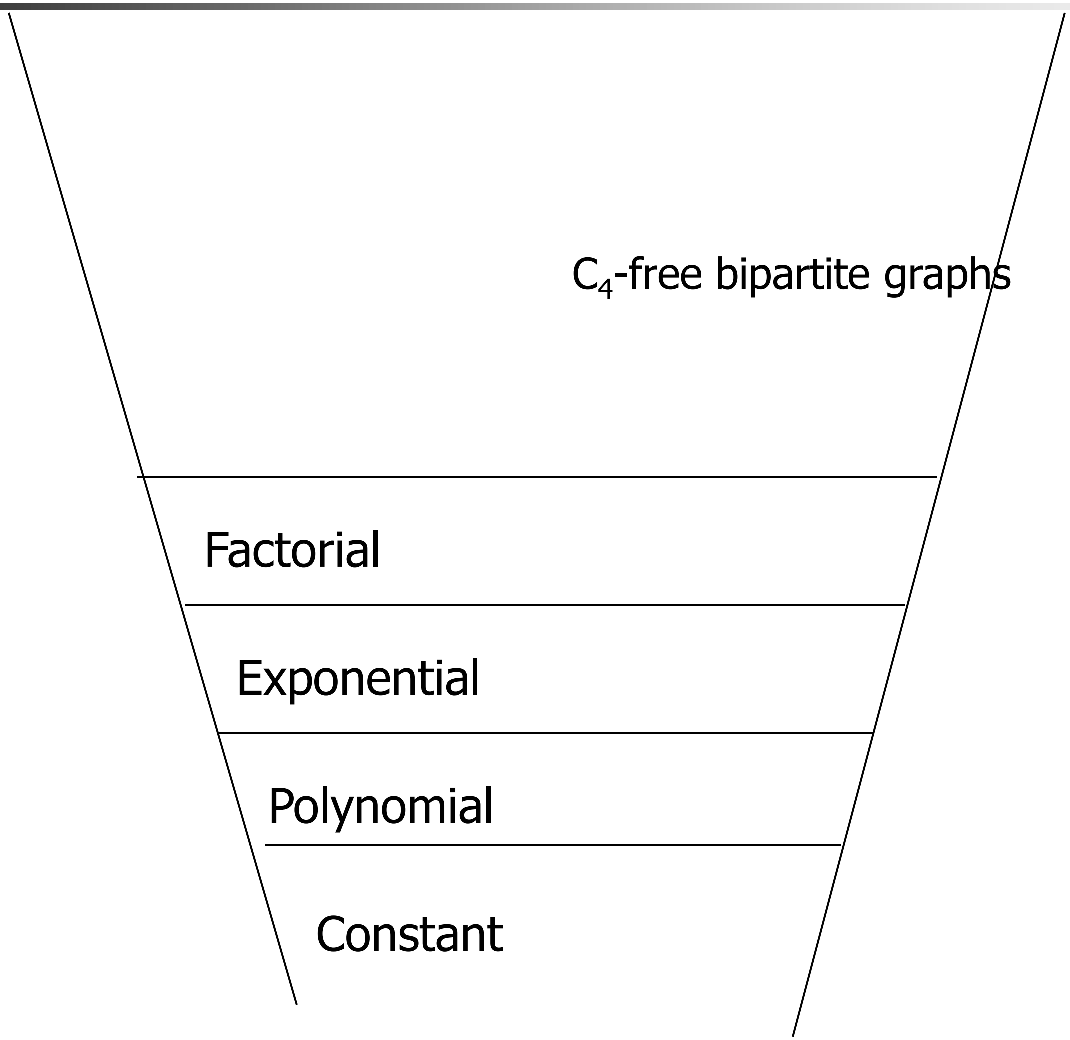
“Smallest” superfactorial class of bipartite graphs



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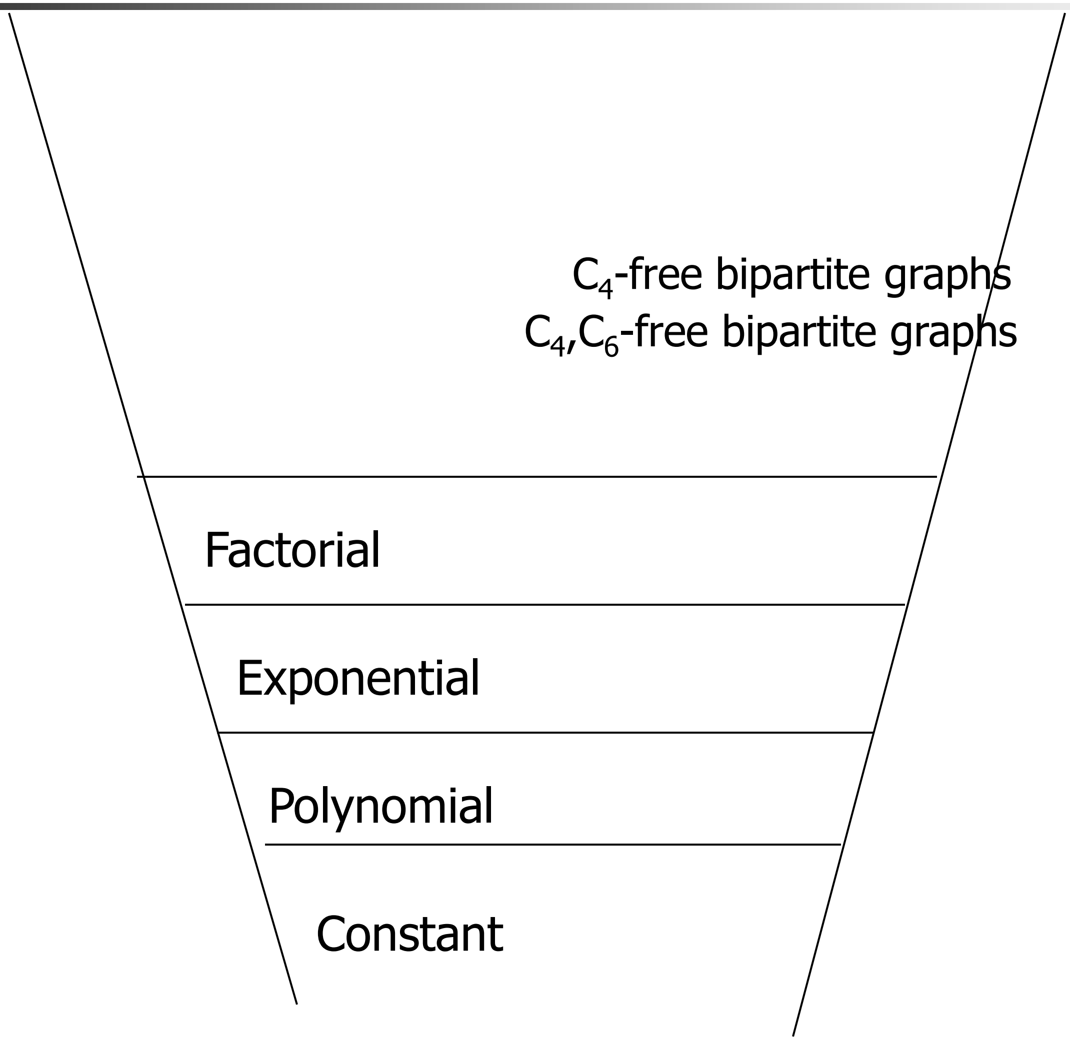

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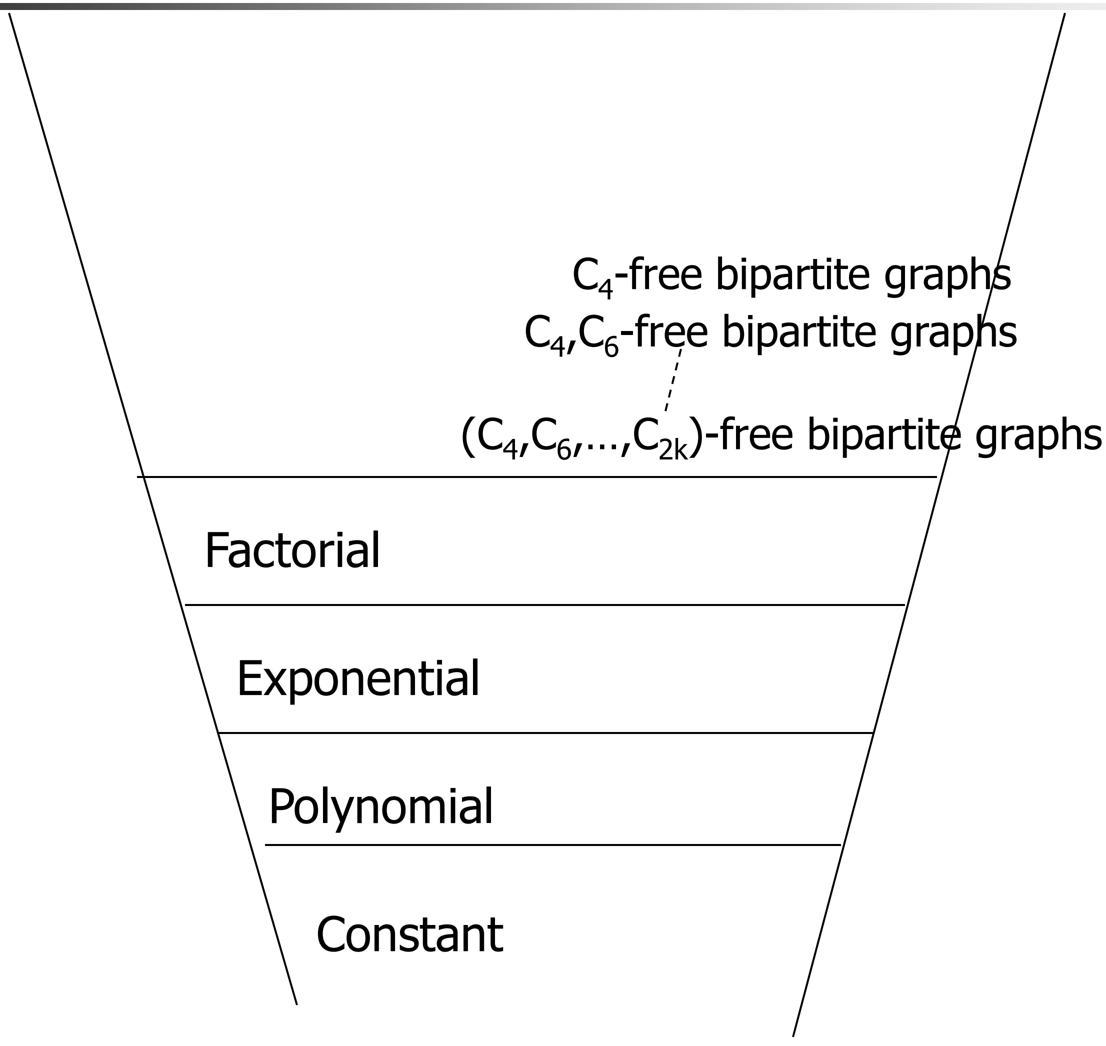

"Smallest" superfactorial class of bipartite graphs

C_4-free bipartite graphs

C_4,C_6-free bipartite graphs

$(C_4,C_6,\ldots,C_{2k})$-free bipartite graphs

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Lazebnik, F.; Ustimenko, V. A.; Woldar, A. J. **A new series of dense graphs of high girth.** *Bull. Amer. Math. Soc. (N.S.)* 32 (1995), no. 1, 73–79.

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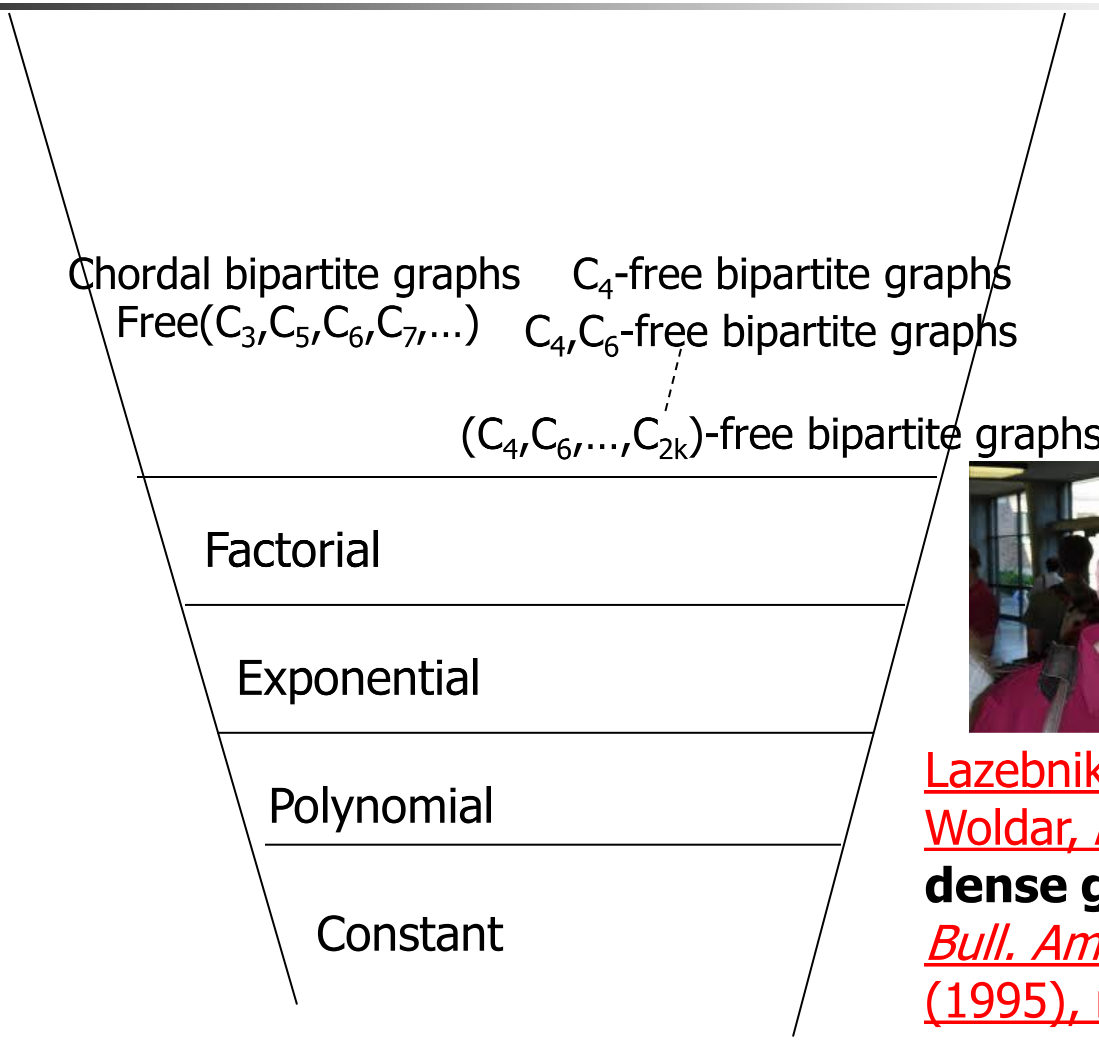

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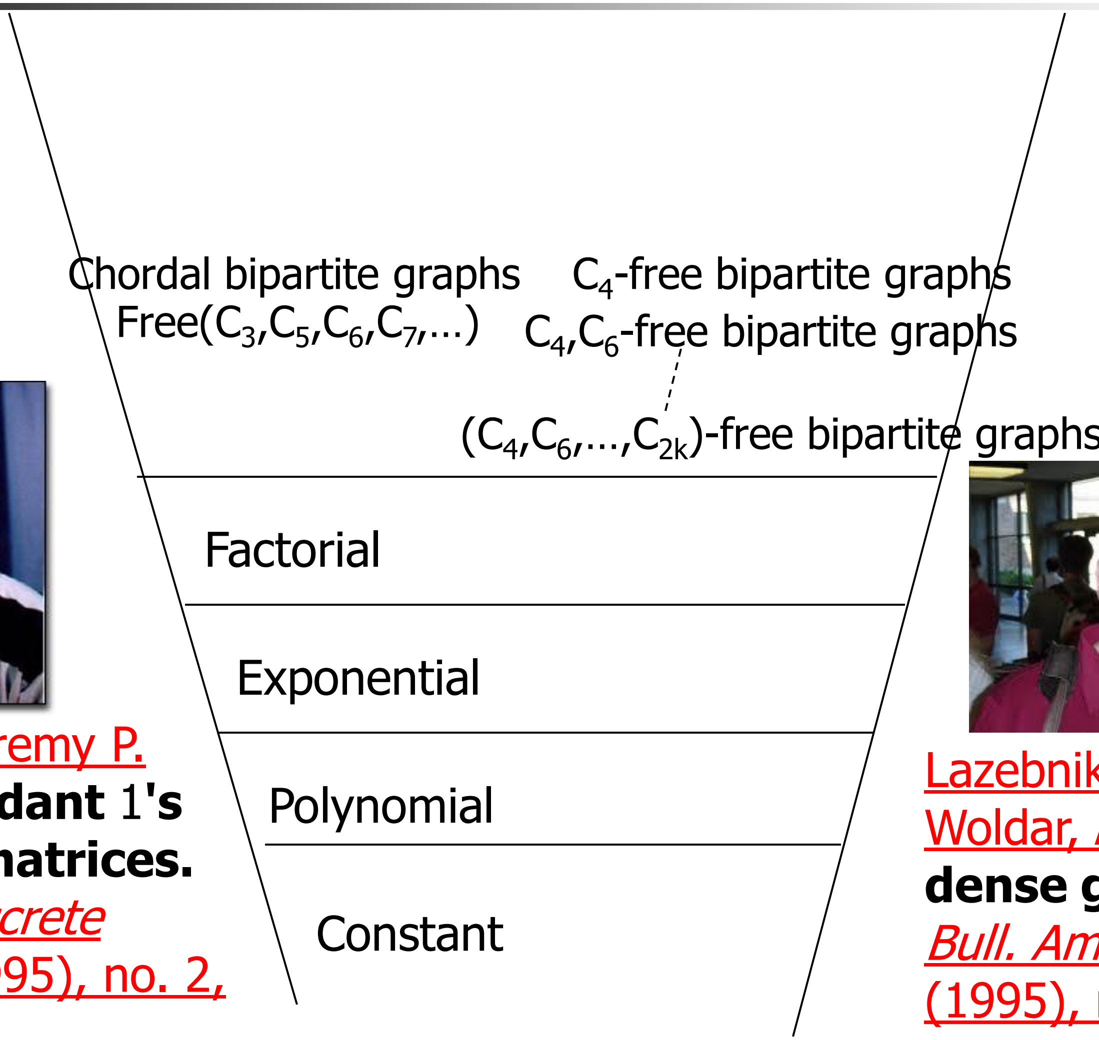

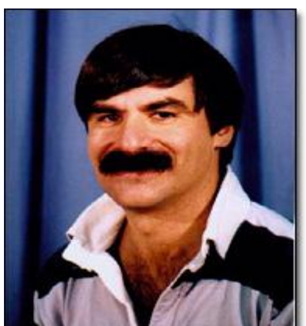

Spinrad, Jeremy P. **Nonredundant 1's in Γ-free matrices.** *SIAM J. Discrete Math.* 8 (1995), no. 2, 251–257

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- all classes of graphs of bounded clique-width

Any hereditary class of graphs of bounded clique-width is at most factorial

P. Allen, V. Lozin, M. Rao **Clique-width and the speed of hereditary properties.** *Electronic J. Combinatorics, 16 (2009), #R35*.

P. Allen

M. Rao

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Partial orders on graphs:

- induced subgraph relation
- subgraph relation
- minor relation
- induced minor relation
- vertex minor relation

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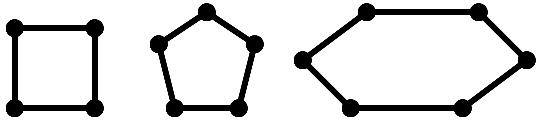
Robertson, Neil; Seymour, P. D. **Graph minors. XX. Wagner's conjecture.** *J. Combin. Theory Ser. B* 92 (2004), no. 2, 325–357

N. Robertson

P. Seymour

Infinite antichains with respect to subgraph and induced subgraph relations



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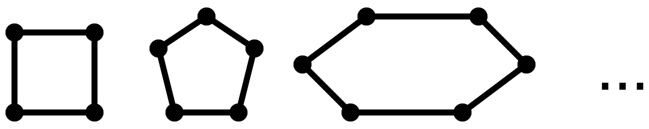

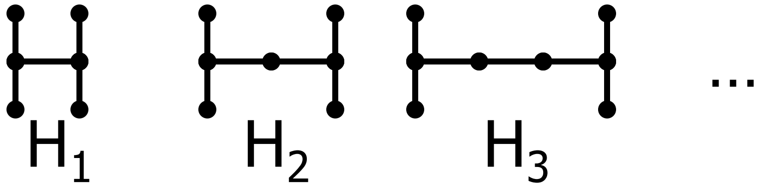

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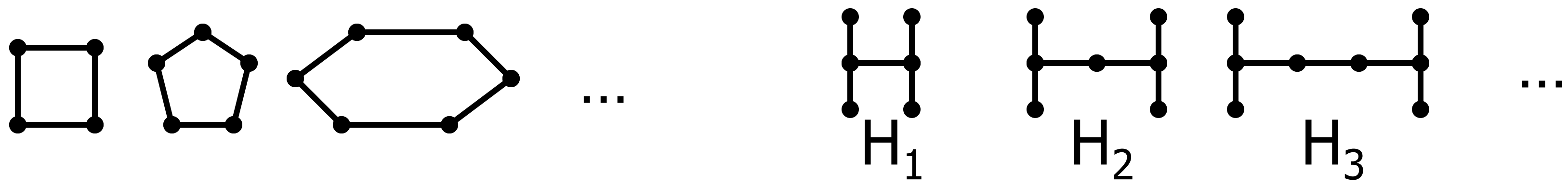

Ding, Guoli **Subgraphs and well-quasi-ordering.**
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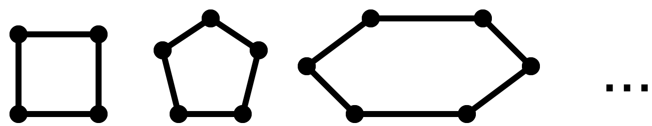

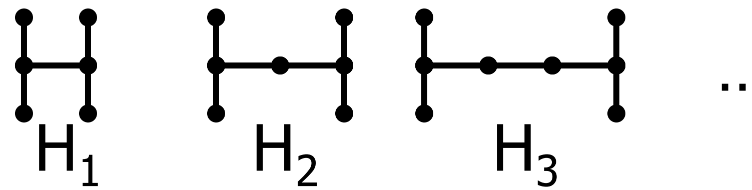

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Ding, Guoli
Position: Professor of Mathematics at LSU
Ph.D.: RUTCOR, Rutgers University 1991. Advisor P.D. Seymour

Hereditary classes of graphs well-quasi-ordered by induced subgraphs

Damaschke, Peter Induced subgraphs and well-quasi-ordering. *J. Graph Theory* 14 (1990), no. 4, 427–435.

Petkovšek, Marko Letter graphs and well-quasi-order by induced subgraphs. *Discrete Math.* 244 (2002), no. 1-3, 375–388.

Korpelainen, N., Lozin, V. Two forbidden induced subgraphs and well-quasi-ordering. *Discrete Math.* 311 (2011), 1813-1822

Korpelainen, N., Lozin, V. Bipartite induced subgraphs and well-quasi-ordering. *J. Graph Theory* 67 (2011), 235–249.

Well-quasi-orderability implies bounded clique-width?

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Daligault, Jean; Rao, Michael; Thomassé, Stéphan Well-quasi-order of relabel functions. *Order* 27 (2010), no. 3, 301–315,

Every well-quasi-ordered hereditary class is at most factorial?

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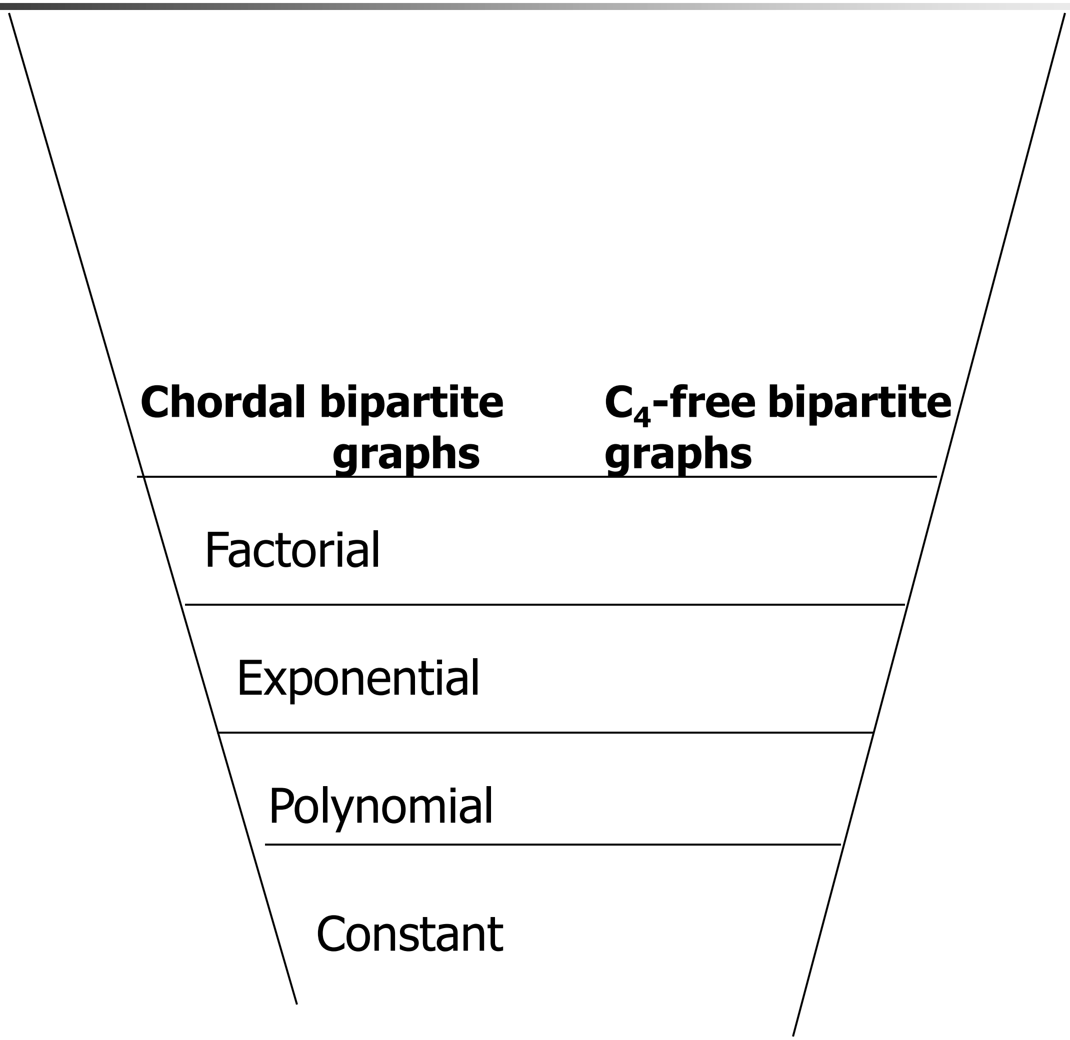

Subclasses of chordal bipartite and C_4-free bipartite graphs without a forest

Theorem. *For any forest F, the classes of F-free chordal bipartite graphs and (F,C_4)-free bipartite graphs are (at most) factorial.*

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Theorem. *For any forest F, the classes of F-free chordal bipartite graphs and (F,C_4)-free bipartite graphs are (at most) factorial.*

Corollary. *Every subclass of chordal bipartite or C_4-free bipartite graphs which is wqo by the induced subgraph relation is (at most) factorial.*

Chordal bipartite graphs
(without a forest)

Let G=(U,V,E) be a bipartite graph with bipartition U,V.

The bipartite adjacency matrix of G is a 0-1 matrix whose rows correspond to U and columns correspond to V.

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Fact. *A bipartite graph is chordal bipartite if and only if its bipartite adjacency matrix has a* Γ*-free ordering.*

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In a Γ-free matrix a 1 entry is called redundant if replacing it with 0 results in a Γ.

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Proof (sketch). W.l.o.g. let F be a rooted tree in which every non-leaf vertex has exactly d children and every leaf vertex is of distance d from the root.

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Let B(1) be the set of the first d non-redundant 1s taken from each column which are not in A(1).

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Let A(i) be the set of the first d non-redundant 1s taken from each row which are not in $A(1) \cup B(1) \cup A(2) \cup B(2) \cup \ldots \cup A(i-1) \cup B(i-1)$.
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1

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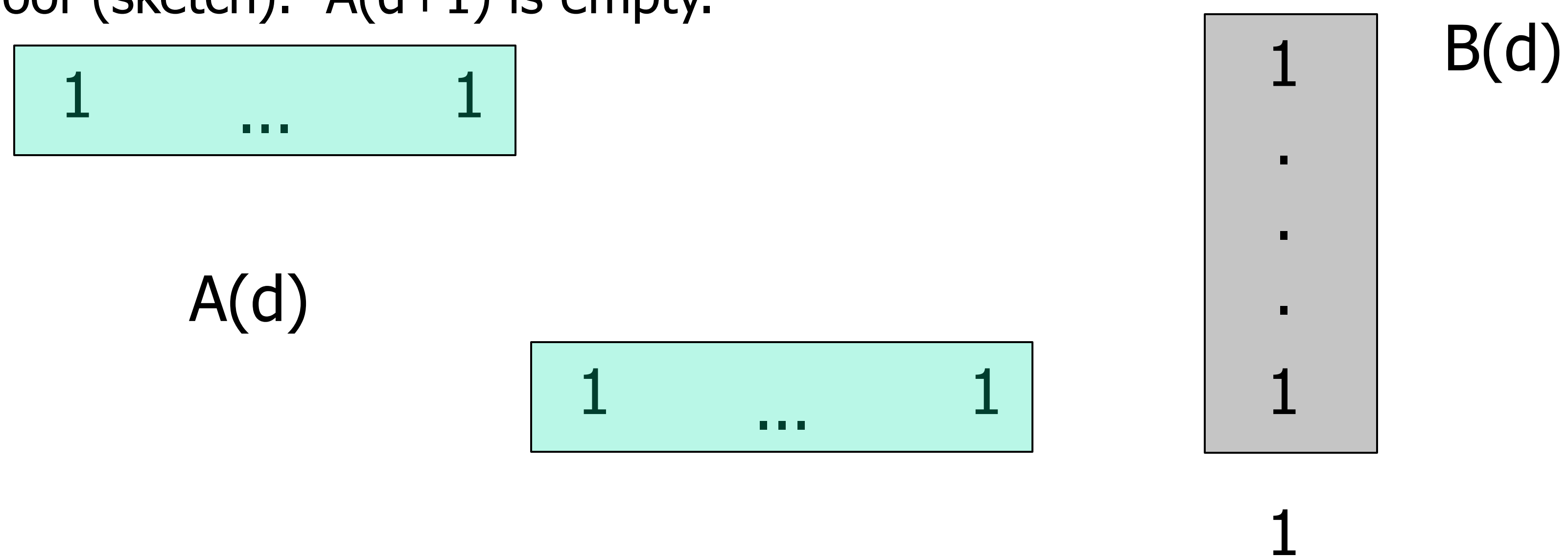

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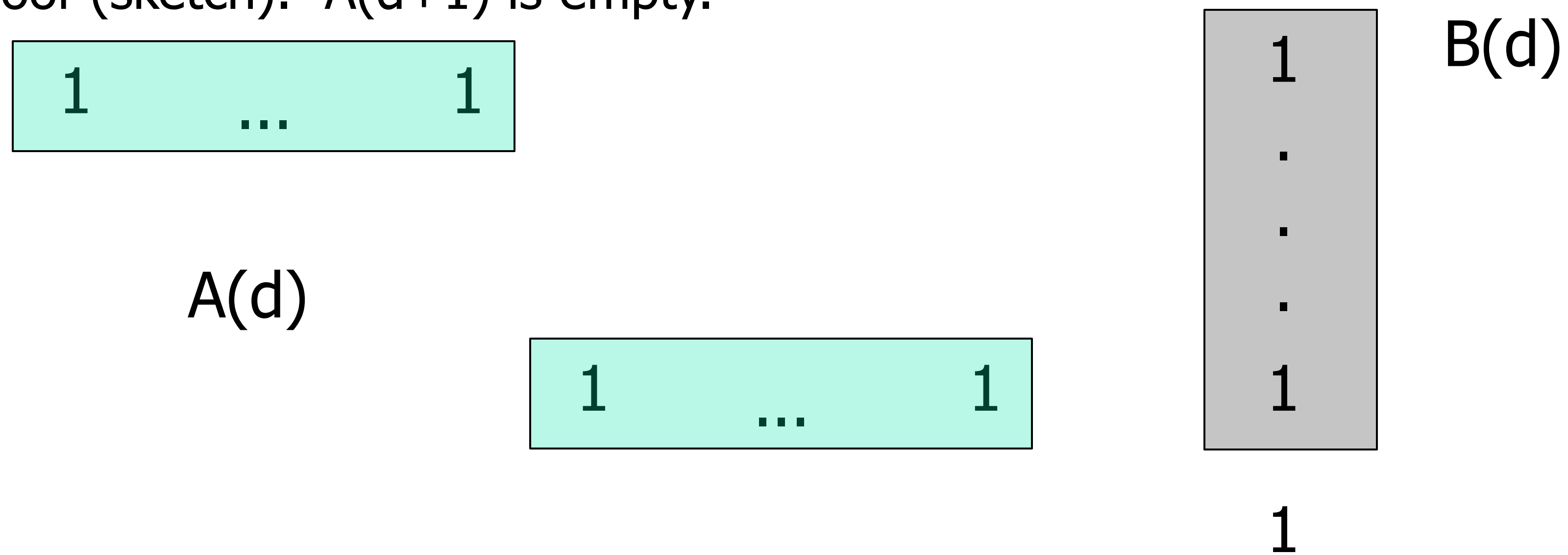

Therefore, the number of non-redundant 1s is at most 2dn.

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Lemma. *Let F be a forest. Then the number of non-redundant 1s in a chordal bipartite F-free graph with n vertices is at most O(n).*

Corollary. *Every F-free chordal bipartite graph can be represented with O(n logn) bits, and therefore, the class of F-free chordal bipartite graphs is at most factorial.*

Subclasses of C_4-free bipartite graphs

Subclasses of C_4-free bipartite graphs without long induced paths

Subclasses of C_4-free bipartite graphs without long induced paths

Corollary. *For every k, the class of (P_k,C_4)-free bipartite graphs is well-quasi-ordered and is of bounded clique-width.*

Subclasses of C_4-free bipartite graphs without long induced paths

Theorem. *For every k,n,p, the class of (P_k,K_n,$K_{p,p}$)-free graphs is well-quasi-ordered and is of bounded clique-width.*

Corollary. *For every k, the class of (P_k,C_4)-free bipartite graphs is well-quasi-ordered and is of bounded clique-width.*

Graphs containing long paths

Theorem. *Every graph containing a long path (not necessarily induced) contains either a big clique or a big induced biclique or a long induced path.*

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Corollary. *For every k,n,p, the exists an s such that the class of (P_k,K_n,$K_{p,p}$)-free graphs is a subclass of the class of graphs containing no P_s as a subgraph (not necessarily induced).*

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Fact. *Any class of graphs closed under taking subgraphs (not necessarily induced) and containing no long paths is of bounded tree-width and well-quasi-ordered.*

Be yourself no matter what they say

Sting

"Englishman in New York"

Be yourself no matter what they say

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Thank you