Induced/distance-k matchings and min-max relations

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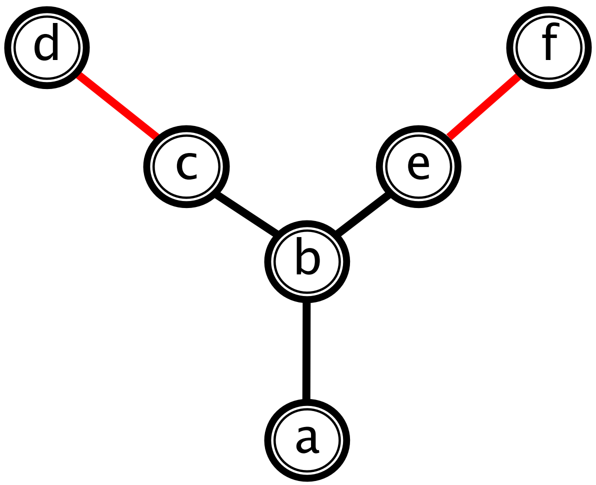
- A distance k-matching in a graph G is a set M $\subseteq$ E(G) such that the distance between any two distinct edges of M is at least k.
- An induced matching, a distance-2 matching, is a matching that is also an induced subgraph.
- im(G) = size of a largest induced matching in G.



Maximum distance-k matching problem (MDkM)

- Given G and k, compute a largest distance-k matching.

- MD2M (finding a largest induced matching) is NP-hard even for bipartite graphs.
  - Stockmeyer and Vazirani (1982)
  - Cameron (1989)

- Suppose G is in a restricted class. Can we solve the problem efficiently ?

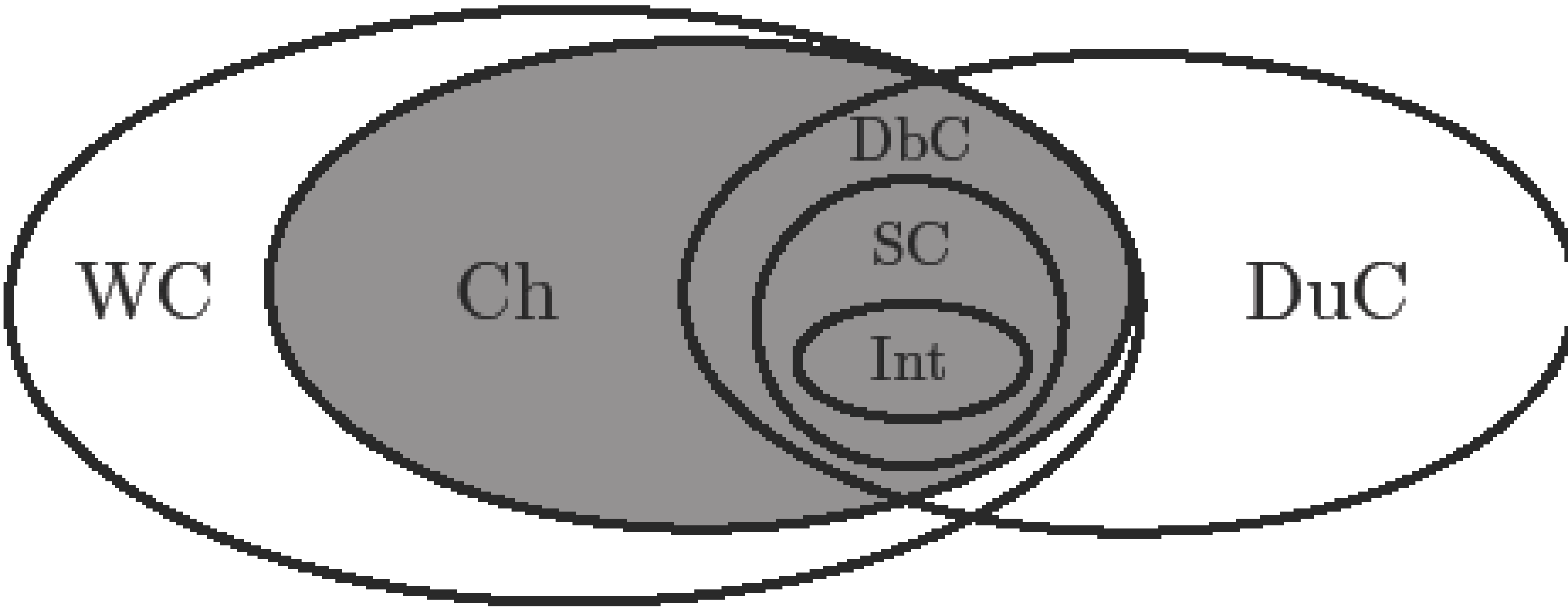
Some classes for which a MD2M can be found in poly time:

- Chordal graphs: Cameron (1989)
- Circular-arc graphs: Golumbic and Laskar (1993)
- Weakly chordal: Cameron, S., Tang (2003)
- AT-free, polygon circle, etc …

- Chordal
  - A graph is chordal if every cycle on four or more vertices has a chord.

- Weakly chordal
  - A graph is weakly chordal if neither the graph nor the complement of the graph has an induced cycle on five or more vertices.

- A graph G is dually chordal provided that the clique hypergraph C(G) is a hypertree.
- When G is dually chordal the intersection graph of the maximal cliques of G (or the line graph of C(G)) is chordal.
- Conversely, when G is chordal the intersection graph of the maximal cliques of G is dually chordal.


DbC
SC
WC
Ch
DuC
Int

(Cameron, 1989)

When G is chordal,
MD2M is in P.

(Brandstadt and Mosca, 2009)

When G is chordal, for $k \geq 1$,
MD(2k)M is in P and
MD3M is NP-hard.

(Busch, Dragan, S. 2010)

When G is weakly chordal, $k \geq 1$,
MD(2k)M is in P and
MD(2k+1)M is NP-hard.

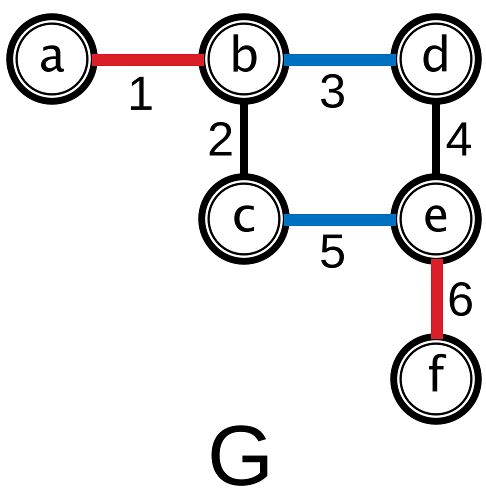
(Busch, Dragan, S. 2010)
When G is dually chordal, $k \geq 1$,
MD(2k+1)M is in P and
MD(2k)M is NP-hard.

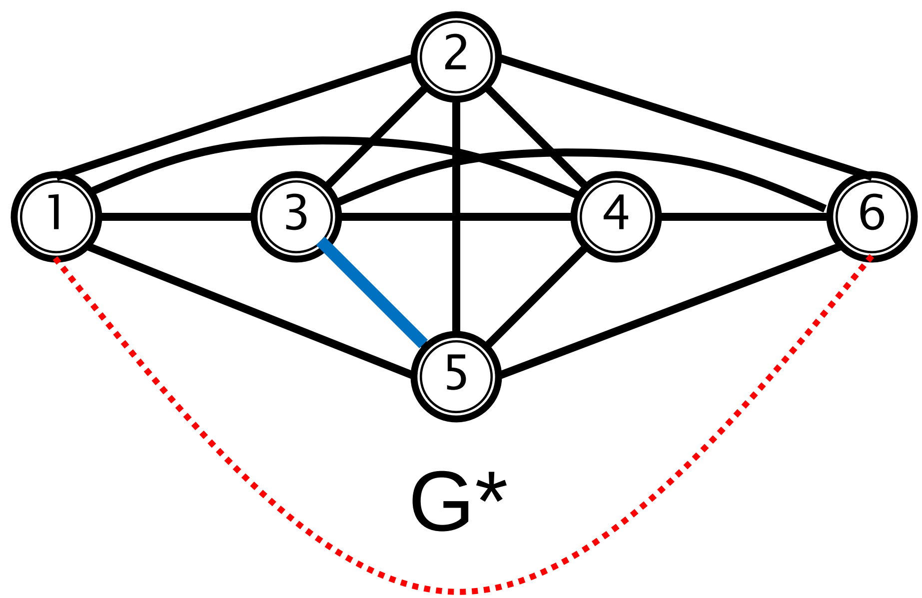
Corollary
When G is doubly chordal, for $k \geq 2$,
MDkM is in P.

(Brandstadt and Mosca, 2009)
When G is strongly chordal, for $k \geq 2$,
MDkM is in P.

Notion of G* (a general idea used):

- Given G = (V, E), consider G* = (E, E*) :
- Vertices e_1 and e_2 are non-adjacent in G* if and only if edges e_1 and e_2 form a $2K_2$ in G.





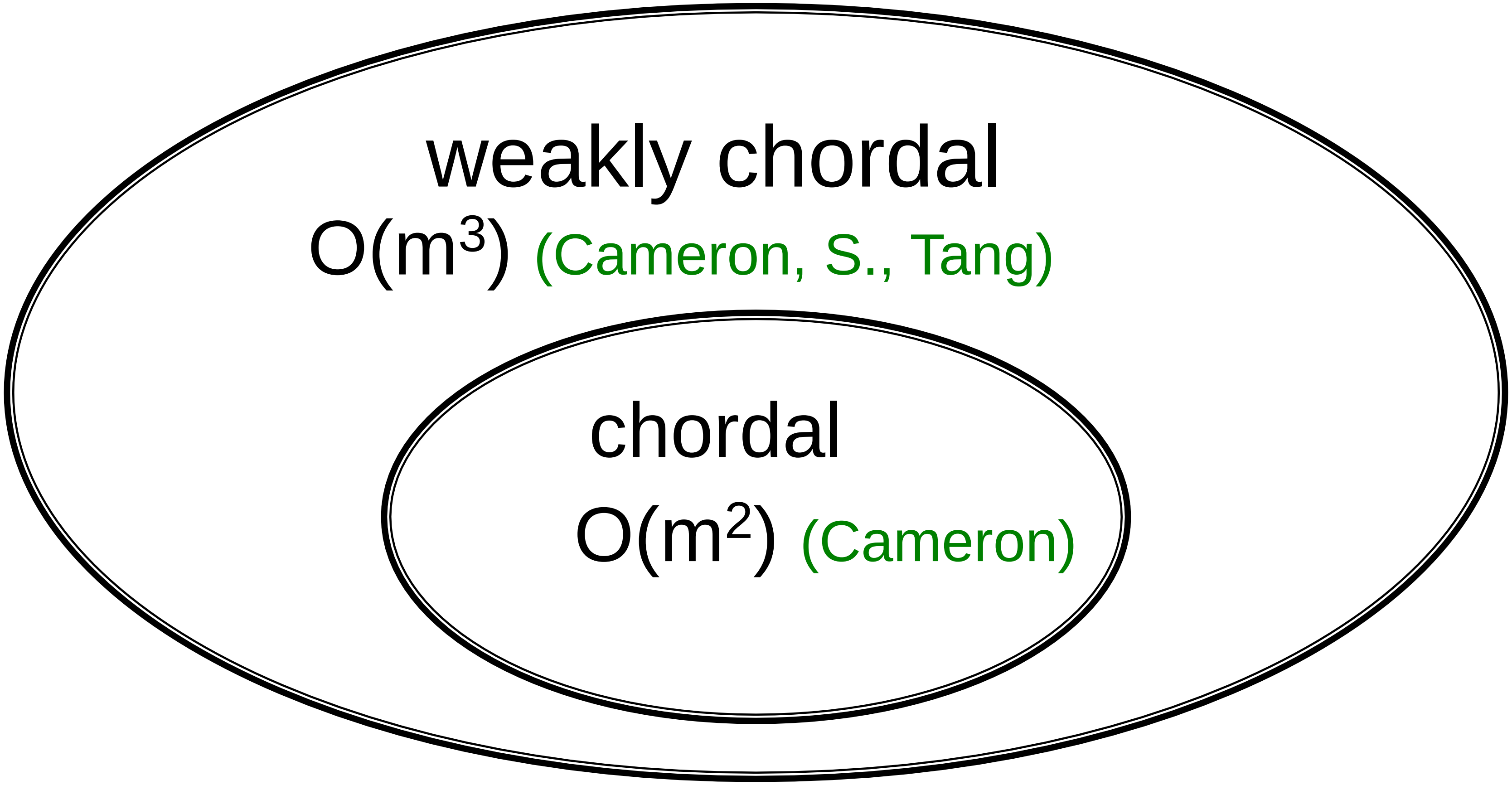
- $G^* = L(G)^2$

- An induced matching in G is an independent set in G^* and vice versa. Therefore, $im(G) = \alpha(G^*)$.

- If $\alpha(G^*)$ can be computed in poly time, then im(G) can be computed in poly time.

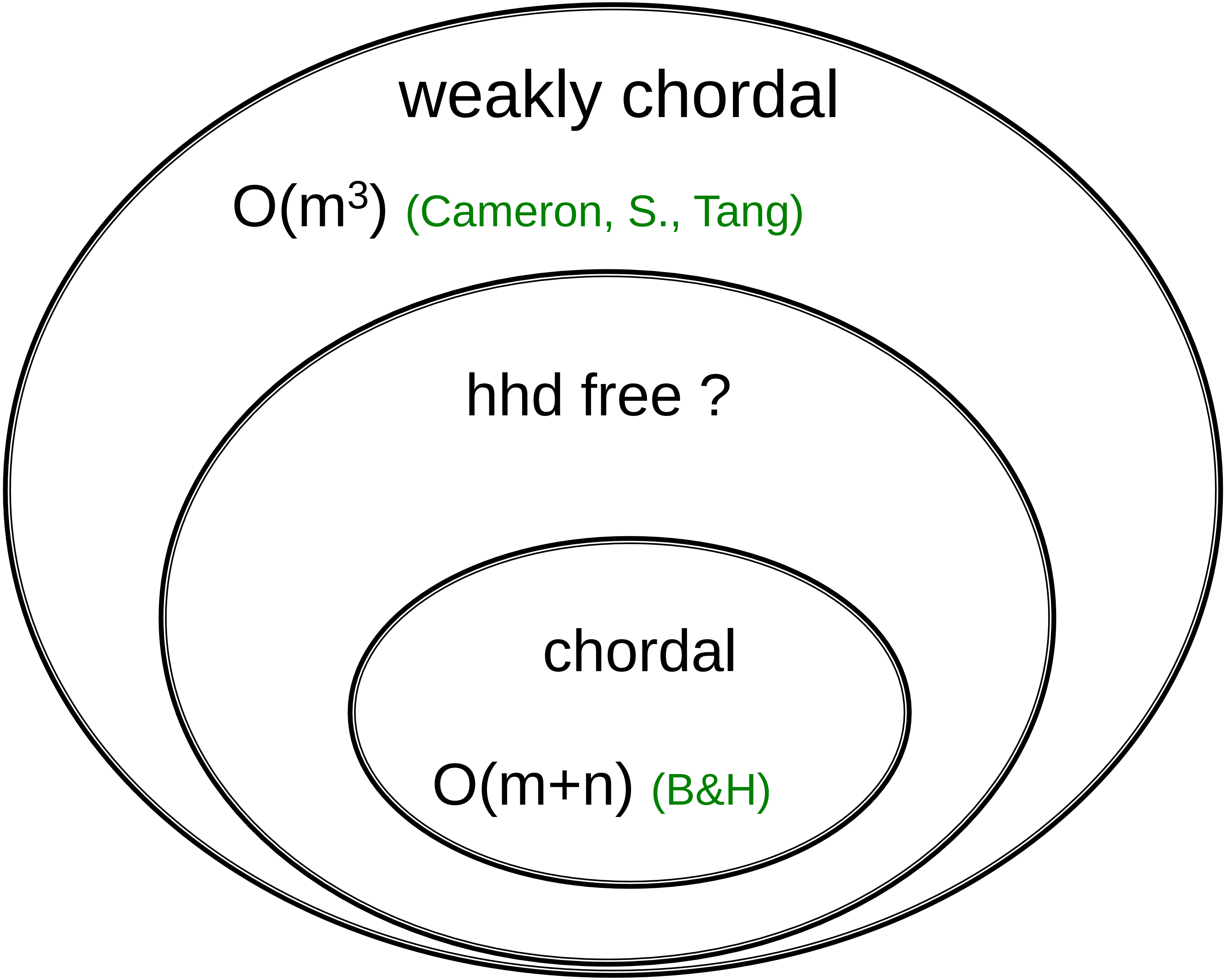
- If G is chordal, then G^* is chordal (Cameron 89).

- If G is weakly chordal, then G^* is weakly chordal (Cameron, S., Tang 03).



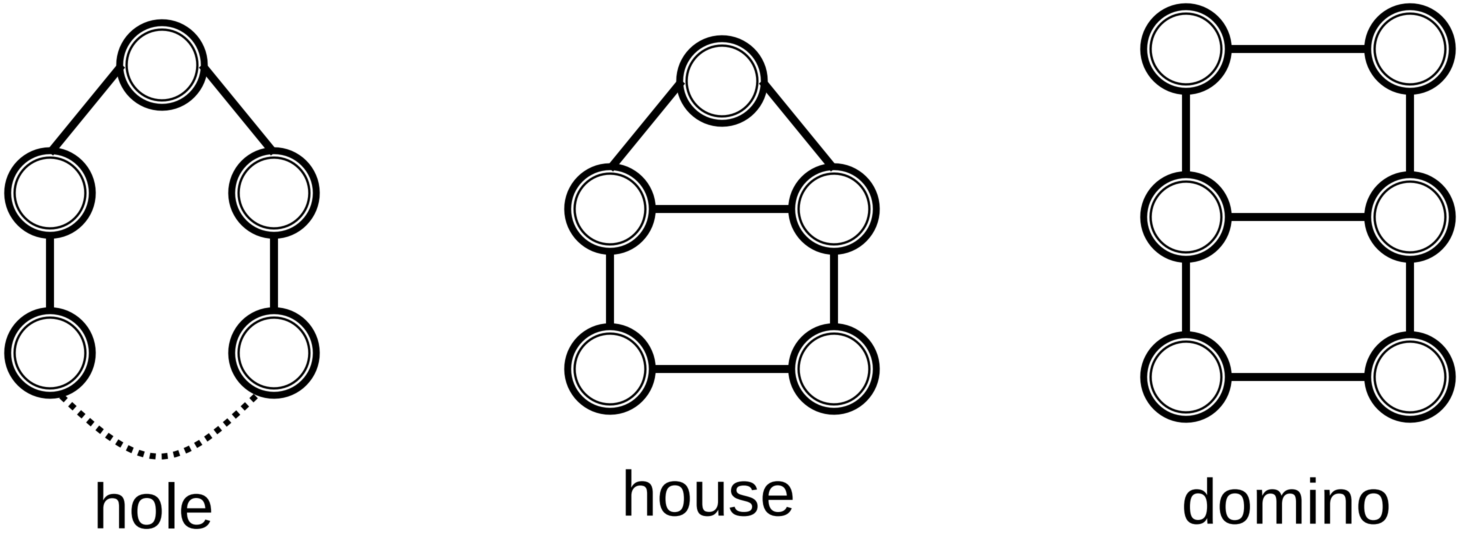
- G* has m vertices and can have $\Omega(m^2)$ edges.
- Construction of G* takes $\Omega(m^2)$ time.
- How do we beat the bottleneck ?

- Theorem (Brandstädt and Hoàng, 2008) A largest induced matching in a chordal graph can be found in linear time.
  - Does not construct G*.
  - Finds an LBFS ordering R of G* from an LBFS ordering of G.
  - Simulates solving for largest independent set in G* on G itself using R.


weakly chordal
O(m³) (Cameron, S., Tang)
hhd free ?
chordal
O(m+n) (B&H)

hhd-free graphs:

- hhd-free graphs do not contain a house, a hole, or a domino



When G is hhd-free, does G^* have special properties ?

Theorem (Krishnamurthy, S. 2011)

If G is hhd-free, then G^* is also hhd-free.

Why would this be useful ?

When G is hhd-free

- G and G' are perfectly orderable (Hoàng and Khouzam, 1988).
  -- so are G* and (G*)'

- LBFS ordering of G is a perfect order of G' (Jamison and Olariu, 1988).
  -- LBFS ordering of G* is a perfect order of (G*)'

- Given a perfect order of G', a largest independent set in G can be found in linear time (Chvátal, Hoàng, Mahadev, and de Werra, 1987).

Corollary

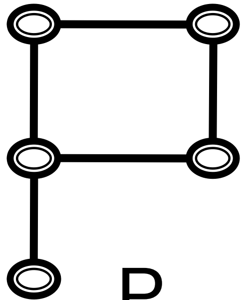
- If G is hhd-free, then the MD2M problem can be solved in $O(m^2)$ time.

  -- Construct G*.

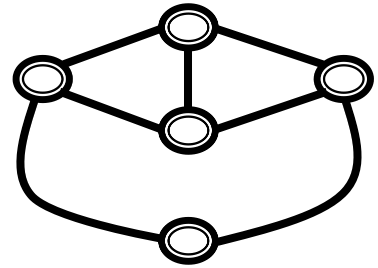
  -- Compute LBFS ordering of G* (a perfect order of (G*)').

  -- Use that to find $\alpha(G^*)$.
- Improvement over algorithm for weakly chordal.

- When G is hhd-free can we do the following (to avoid building G*) ?
  - Obtain perfect order of (G*)' only looking at G.
  - Find a largest independent set in G* only by looking at G.



P



diamond-ring

Theorem (Krishnamurthy, S. 2011)

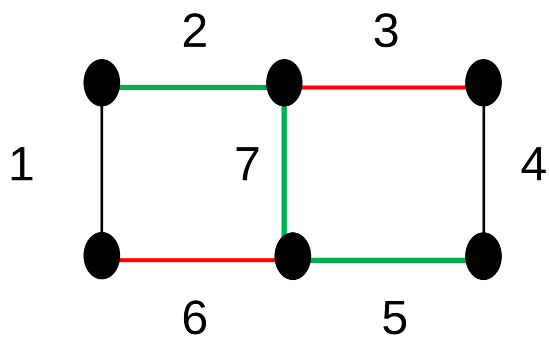
When G is a hhd-free graph without a P or a diamond-ring,

- an LBFS ordering of G* can be found from an LBFS ordering of G in linear time, and
- MD2M an be solved in linear time.

-- this is a more general class than chordal.

Min-max relations

- H is a chain subgraph of bipartite graph G if H is a subgraph of G and H has no $2K_2$.



Red – not a chain subgraph
Green – a chain subgraph

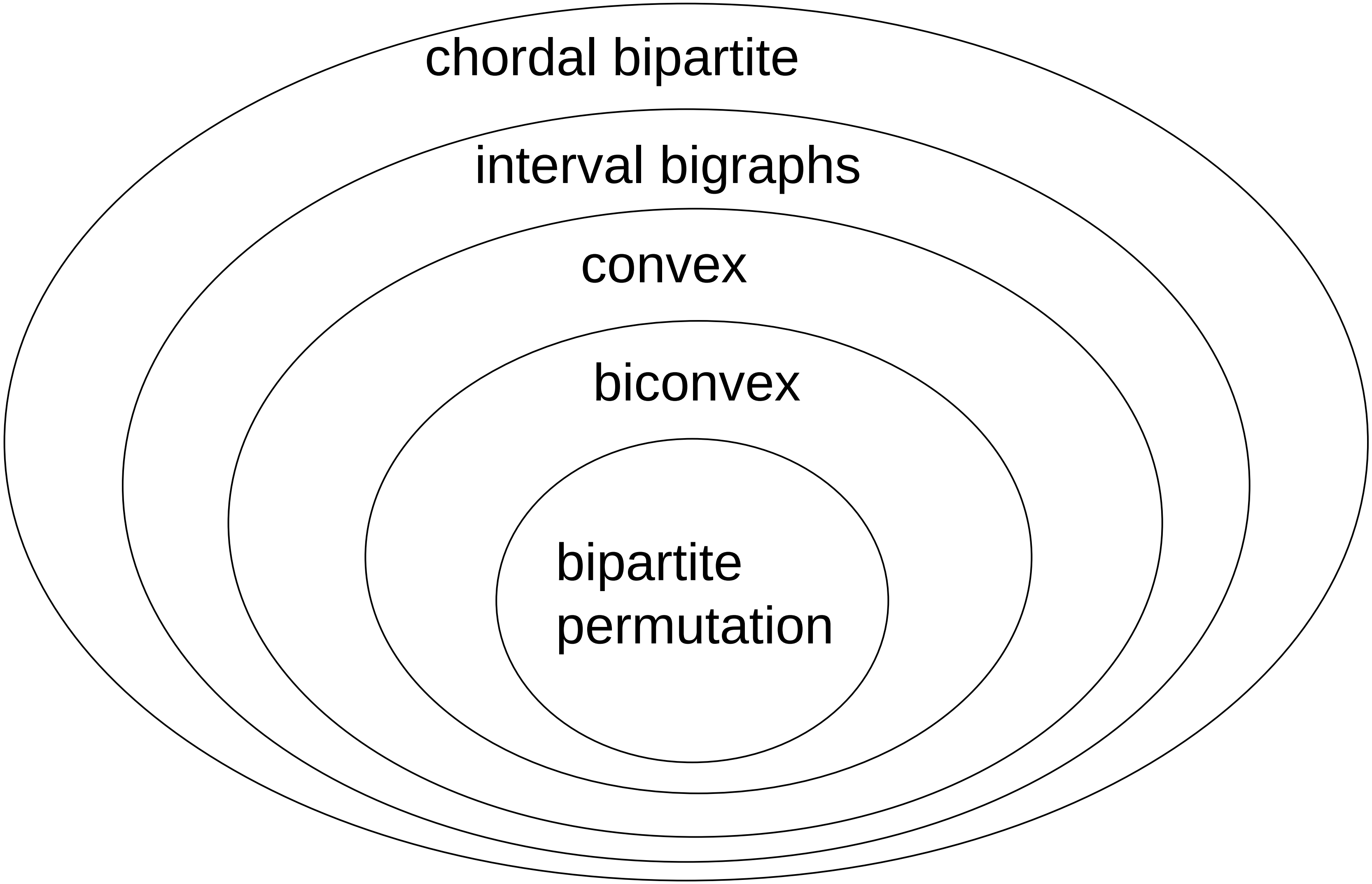
ch(G) = minimum number of chain subgraphs of G needed to cover edges of G.

Theorem (Yannakakis 81) Computing ch(G) is NP-hard.

Clearly for any bipartite graph G, $im(G) \leq ch(G)$.

Theorem (Cozzens, Leibowitz 1984) ch(G) and im(G) can differ arbitrarily.

A hierarchy of bipartite graph classes



Defintion

$G = (X, Y, E)$ is a bipartite graph.

G is convex if Y can be ordered so that for any v in X, neighbors of v occur consecutively in the ordering.

Theorem (Yu, Chen, Ma 1998) When bipartite G is convex, $im(G) = ch(G)$ and a minimum chain subgraph cover can be found in $O(m^2)$ time.

Theorem (Brandstädt, Eschen, S. 2005) Their algorithm can be implemented to run in $O(n^2)$ time.

Defintion

Bipartite graph G is chordal bipartite if G has no induced cycles on six or more vertices.

- connections to totally balanced matrices, Γ–free ordering of matrices (Anstee, Farber, Hoffman, Kolen, Sakarovitch, Lubiw).

Theorem (Abueida, Busch, S. 2008)

For a chordal bipartite graph G, im(G) = ch(G) and a minimum chain subgraph cover can be found in poly time.

Theorem (Cameron 1989)

For a chordal graph G, $\mathrm{im}(G)$ = size of a minimum cover with split subgraphs and such a minimum cover can be found in poly time.

Common theme ?

A chain graph is co-chordal.
A split graph is co-chordal.
A chordal bipartite graph is weakly chordal.
A chordal graph is weakly chordal.

Theorem (Busch, Dragan, S. 2010)

For a weakly chordal graph G, im(G) = size of a minimum cover with co-chordal subgraphs and such a minimum cover can be found in poly time.

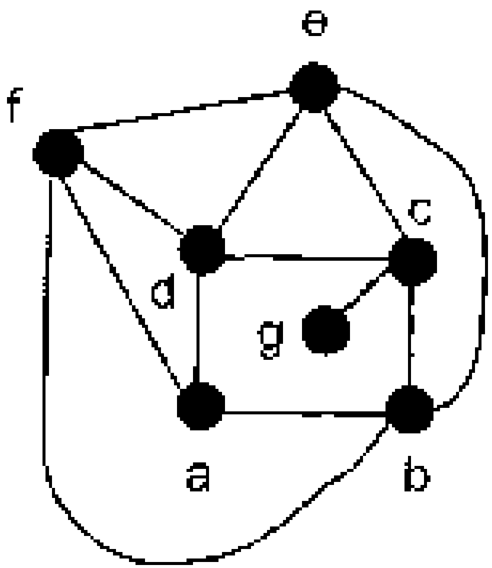
Corollary

The “chordal dimension” of a weakly chordal graph can be found in poly time.

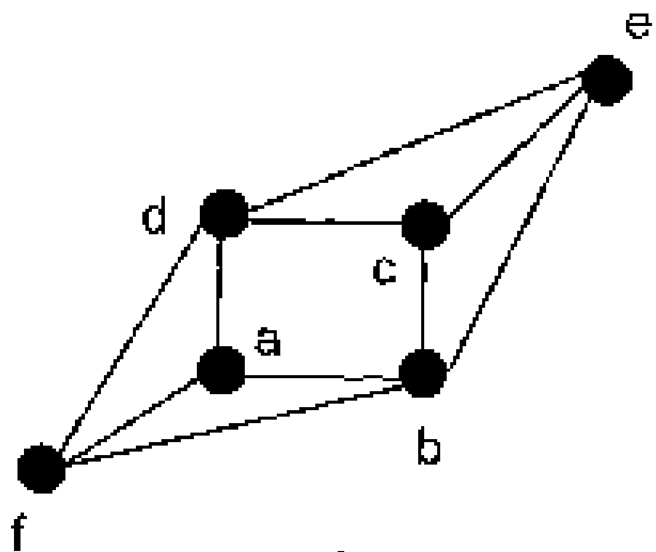
A distance-k vertex cover in G is a set of vertices such that every edge in G is within distance k from at least one vertex in the set.

Theorem (Busch, Dragan, S. 2010)

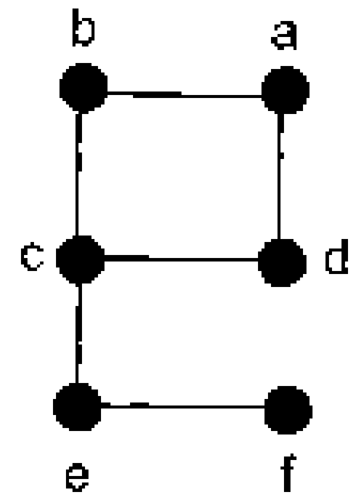
For $k \geq 1$, if G is dually chordal, then the size of a largest distance-$2k+1$ matching equals the size of a smallest distance-k vertex cover and they can be found in polynomial time.



G_1



G_2



G_3

How to prove ?

- Prove by induction on number of edges that a maximal clique of G^* maps to a maximal co-chordal subgraph of G.

Tools ?

- (Eschen, S.) When weakly chordal G has a $2K_2$, it has co-pair e and co-pair f that form a $2K_2$.
- (Spinrad, S.) G–e and G–f are weakly chordal.
- $(G-e)^* = G^* \setminus e$
- $(G-f)^* = G^* \setminus f$

Approcah: some relevant questions

- When G is hhd-free, does G* have any special properties ?
- If so, could one use those properties to find α(G*) without constructing G* (and working on G only) ?

Min-max relations

- For any graph G, $\mathrm{im}(G) = \alpha(G^*)$.
- Further, when G^* is perfect, $\alpha(G^*) = \theta(G^*)$.
- Can we translate $\mathrm{im}(G) = \alpha(G^*) = \theta(G^*)$ back to G ?

  -- not obvious