Partitions of Chordal Graphs

Pavol Hell, Simon Fraser University

GROW, October 29, 2011
Co-authors

- Tomás Feder
- Shekoofeh Nekooei Rizi
Graph Partitions

Parts and their connections are constrained. Assign vertices to parts so that constraints are satisfied.

Constraints on parts:
- Independent set
- Clique
- Unrestricted

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Parts and their connections are constrained

Assign vertices to parts so that constraints are satisfied
Parts and their connections are constrained

Assign vertices to parts so that constraints are satisfied

Constraints on parts

- Independent set
- Clique
- Unrestricted
Graph Partitions

Parts and their connections are constrained
Assign vertices to parts so that constraints are satisfied

Constraints between parts
- No edges
- All edges
- Unrestricted
Graph Partitions

Parts and their connections are constrained
Assign vertices to parts so that constraints are satisfied

Partition into $k$ parts

A $k \times k$ symmetric matrix $M$ over $0, 1, \ast$
- No edges = 0
- All edges = 1
- Unrestricted = $\ast$
A $k$ by $k$ symmetric matrix $M$ over $0, 1, *$

An $M$-partition of $G$
A $k$ by $k$ symmetric matrix $M$ over $0, 1, *$

An $M$-partition of $G$

A partition $V_1, V_2, \ldots, V_k$ of $V(G)$ with

- $V_i$ a clique if $M(i, i) = 1$, and independent set if $M(i, i) = 0$
- all edges between $V_i$ and $V_j$ present if $M(i, j) = 1$, and absent if $M(i, j) = 0$
A $k$ by $k$ symmetric matrix $M$ over $0, 1, *$

An $M$-partition of $G$

A partition $V_1, V_2, \ldots, V_k$ of $V(G)$ with

- $V_i$ a clique if $M(i, i) = 1$, and independent set if $M(i, i) = 0$
- all edges between $V_i$ and $V_j$ present if $M(i, j) = 1$, and absent if $M(i, j) = 0$

The $M$-partition problem

Decide if input graph $G$ has an $M$-partition
Matrix $M$

$$
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0
\end{pmatrix}
$$

Is $G$ 3-colourable?

NP-complete
Example

Matrix $M$

\[
\begin{pmatrix}
  0 & \ast & \ast \\
  \ast & 0 & \ast \\
  \ast & \ast & 0
\end{pmatrix}
\]

The $M$-partition problem

Is $G$ 3-colourable?

NP-complete
Matrix $M$

$$
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
* & 0 & 0
\end{pmatrix}
$$
Example

Matrix $M$

\[
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
* & 0 & 0
\end{pmatrix}
\]

The $M$-partition problem

Does $G$ admit a homomorphism to $P_3$?

Polynomial time solvable
Example

Matrix $M$

\[
\begin{pmatrix}
0 & * & * \\
* & 0 & 0 \\
* & 0 & 0
\end{pmatrix}
\]

$M$-partition problems
Generalize graph colouring and homomorphism problems
Matrix $M$

\[
\begin{pmatrix}
1 & * & * \\
* & * & 0 \\
* & 0 & *
\end{pmatrix}
\]

Does $G$ have a clique cutset? Polynomial time solvable (Whitesides 1981, Tarjan 1985)

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Matrix $M$

\[
\begin{pmatrix}
1 & * & * \\
* & * & 0 \\
* & 0 & *
\end{pmatrix}
\]

An $M$-partition of $G$ (with all parts nonempty)
Does $G$ have a clique cutset?

Polynomial time solvable (Whitesides 1981, Tarjan 1985)
Variants of $M$-partition problems

Surjective partitions
Each part is non-empty
Variants of $M$-partition problems

Surjective partitions
Each part is non-empty

$M$ not necessarily symmetric
Partitioning digraphs $G$
### Variants of $M$-partition problems

<table>
<thead>
<tr>
<th><strong>Surjective partitions</strong></th>
<th>Each part is non-empty</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$M$ not necessarily symmetric</strong></td>
<td>Partitioning digraphs $G$</td>
</tr>
<tr>
<td><strong>Vertices of the graph $G$ are equipped with lists</strong></td>
<td>Each vertex placed in a part on its list</td>
</tr>
</tbody>
</table>
### Variants of $M$-partition problems

<table>
<thead>
<tr>
<th>Description</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Surjective partitions</strong></td>
<td>Each part is non-empty</td>
</tr>
<tr>
<td><strong>$M$ not necessarily symmetric</strong></td>
<td>Partitioning digraphs $G$</td>
</tr>
<tr>
<td><strong>Vertices of the graph $G$ are equipped with lists</strong></td>
<td>Each vertex placed in a part on its list</td>
</tr>
<tr>
<td><strong>Edge-coloured complete graphs $G$</strong></td>
<td>$M$ records which colours are allowed</td>
</tr>
</tbody>
</table>
Study of perfect graphs

- Split graphs
  - Polynomial time recognition by Foldes-Hammer 1973
  - Skew cutset? Chvátal 1973
  - Quasi-polynomial with lists, Feder-Hell-Klein-Motwani 2003
  - Polynomial with lists, Figueiredo-Klein-Kohayakawa-Reed 2007

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Study of perfect graphs

- Split graphs
- Polynomial time recognition by Foldes-Hammer (1973)
- Skew cutset? by Chvátal (1973)
- Quasi-polynomial with lists by Feder, Hell, Klein, Motwani (2003)
- Polynomial with lists by Figueiredo, Klein, Kohayakawa, Reed (2007)

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Study of perfect graphs

Split graphs
Polynomial time recognition by Foldes-Hammer 1973
Study of perfect graphs

Split graphs
Polynomial time recognition by Foldes-Hammer 1973

Skew cutset? Chvátal 1973
Study of perfect graphs

Split graphs
Polynomial time recognition by Foldes-Hammer 1973

Skew cutset? Chvátal 1973
Quasi-polynomial with lists, Feder-Hell-Klein-Motwani 2003
Study of perfect graphs

Split graphs
Polynomial time recognition by Foldes-Hammer 1973

Skew cutset? Chvátal 1973
Quasi-polynomial with lists, Feder-Hell-Klein-Motwani 2003
Polynomial with lists, Figueiredo-Klein-Kohayakawa-Reed 2007
Given $M$

Is the $M$-partition problem NP-complete or polynomial time solvable?
Complexity of $M$-partition

Given $M$

Is the $M$-partition problem NP-complete or polynomial time solvable?

In the digraph variant, if $M$ has no 1’s

The Dichotomy Conjecture of CSP: YES
### Given $M$

Is the $M$-partition problem NP-complete or polynomial time solvable?

### In the digraph variant, if $M$ has no 1’s

The Dichotomy Conjecture of CSP: YES  
Each $M$-partition problem is NP-complete or solvable in polynomial time
Complexity of $M$-partition

Given $M$

Is the $M$-partition problem NP-complete or polynomial time solvable?

In the digraph variant, if $M$ has no 1’s

The Dichotomy Conjecture of CSP: YES
Each $M$-partition problem is NP-complete or solvable in polynomial time

Feder-Hell 2006

Each $M$-partition problem is NP-complete or solvable in quasi-polynomial time
Complexity of $M$-partition for restricted graphs

Given $M$ is the $M$-partition problem for perfect graphs NP-complete or polynomial time solvable? Feder-Hell 2006

Each constraint satisfaction problem is polynomial time equivalent to some $M$-partition problem for perfect graphs ($\forall H \exists M$ with $\text{CSP}(H) \sim P_M$-partition). Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Given $M$

Is the $M$-partition problem for perfect graphs NP-complete or polynomial time solvable?
Complexity of $M$-partition for restricted graphs

Given $M$

Is the $M$-partition problem for perfect graphs NP-complete or polynomial time solvable?

Feder-Hell 2006

Each constraint satisfaction problem is polynomial time equivalent to some $M$-partition problem for perfect graphs
Given $M$

Is the $M$-partition problem for perfect graphs NP-complete or polynomial time solvable?

Feder-Hell 2006

Each constraint satisfaction problem is polynomial time equivalent to some $M$-partition problem for perfect graphs

$(\forall H \exists M \text{ with } \text{CSP}(H) \sim_P M\text{-partition})$
Cographs

Every $M$-partition problem can be solved in linear time
Complexity of $M$-partition for restricted graphs

**Cographs**

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$
Complexity of $M$-partition for **restricted** graphs

**Cographs**

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

**Chordal graphs**

Polynomial
Gavril 1971
Complexity of $M$-partition for restricted graphs

**Cographs**

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

**Chordal graphs**

$$
\begin{pmatrix}
0 & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 0 & * \\
* & * & * & * & 0 \\
\end{pmatrix}
$$
Complexity of $M$-partition for restricted graphs

**Cographs**

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000

Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

**Chordal graphs**

\[
\begin{pmatrix}
0 & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 0 & * \\
* & * & * & * & 0
\end{pmatrix}
\]

Polynomial

Gavril 1971
Complexity of $M$-partition for restricted graphs

Cographs

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

Chordal graphs

$\begin{pmatrix}
1 & * & * & * & * & * \\
* & 1 & * & * & * & * \\
* & * & 1 & * & * & * \\
* & * & * & 1 & * & * \\
* & * & * & * & 1 & *
\end{pmatrix}$

Polynomial
Gavril 1971
Complexity of $M$-partition for restricted graphs

Cographs

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

Chordal graphs

\[
\begin{pmatrix}
0 & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 1 & * \\
* & * & * & * & 1
\end{pmatrix}
\]
Complexity of $M$-partition for restricted graphs

Cographs

Every $M$-partition problem can be solved in linear time

Courcelle-Makowsky-Rotics 2000
Monadic second order logic description of $M$-partition: $\exists A \exists B \ldots$

Chordal graphs

\[
\begin{pmatrix}
0 & * & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 1 & * \\
* & * & * & * & 1
\end{pmatrix}
\]

Polynomial
(Gimbel-Kratsch-Stewart 1994)
Chordal graphs

Is every $M$-partition problem polynomial?

Feder-Hell-Klein-Nogueira-Protti 2005
Complexity of $M$-partition for *chordal* graphs

**Chordal graphs**

Is every $M$-partition problem polynomial?

**Chordal graphs**

If $M$ has 0 diagonal then the $M$-partition problem is polynomial.

Feder-Hell-Klein-Nogueira-Protti 2005
Chordal graphs
Is every $M$-partition problem polynomial?

If $M$ has 0 diagonal then the $M$-partition problem is polynomial

Feder-Hell-Klein-Nogueira-Protti 2005

Example

\[
\begin{pmatrix}
0 & * & * & 1 & 1 \\
* & 0 & 1 & 1 & * \\
* & 1 & 0 & * & 0 \\
1 & 1 & * & 0 & * \\
1 & * & 0 & * & 0 \\
\end{pmatrix}
\]
Chordal graphs

Is every $M$-partition problem polynomial?

Chordal graphs

If $M$ has 1 diagonal then the $M$-partition problem is polynomial

Feder-Hell-Klein-Nogueira-Protti 2005

Example

\[
\begin{pmatrix}
1 & * & * & 1 & 1 \\
* & 1 & 1 & 1 & * \\
* & 1 & 1 & * & 0 \\
1 & 1 & * & 1 & * \\
1 & * & 0 & * & 1
\end{pmatrix}
\]
### Chordal graphs

Is every $M$-partition problem polynomial?

### A crossed matrix

$$
\begin{pmatrix}
0 & * & * & 1 & 1 \\
* & 0 & 1 & 1 & 0 \\
* & 1 & 0 & * & 0 \\
1 & 1 & * & 1 & * \\
1 & 0 & 0 & * & 1 \\
\end{pmatrix}
$$

Each 0 and each 1 lies in a red row or column without *
Complexity of $M$-partition for **chordal** graphs

**Chordal graphs**

Is every $M$-partition problem polynomial?

**A crossed matrix**

$$
\begin{pmatrix}
0 & \ast & \ast & 1 & 1 \\
\ast & 0 & 1 & \ast & 0 \\
\ast & 1 & 0 & \ast & 0 \\
1 & \ast & \ast & 1 & \ast \\
1 & 0 & 0 & \ast & 1 \\
\end{pmatrix}
$$

Each 0 and each 1 lies in a red row or column without *
Complexity of $M$-partition for chordal graphs

Chordal graphs

Is every $M$-partition problem polynomial?

Not a crossed matrix

\[
\begin{pmatrix}
0 & * & * & 1 & 1 \\
* & 0 & 1 & 1 & * \\
* & 1 & 0 & * & 0 \\
1 & 1 & * & 1 & * \\
1 & * & 0 & * & 1
\end{pmatrix}
\]

Each 0 and each 1 lies in a red row or column without *
Complexity of $M$-partition for **chordal** graphs

**Chordal graphs**

Is every $M$-partition problem polynomial?

**A crossed matrix**

$$\begin{pmatrix}
0 & * & * & 1 & 1 \\
* & 0 & 1 & 1 & * \\
* & 1 & 0 & * & 0 \\
1 & 1 & * & 0 & * \\
1 & * & 0 & * & 0
\end{pmatrix}$$

Each 0 and each 1 lies in a red row or column without *
Complexity of $M$-partition for chordal graphs

**Chordal graphs**

Is every $M$-partition problem polynomial?

**A crossed matrix**

\[
\begin{pmatrix}
0 & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 1 & * \\
* & * & * & * & 1 \\
\end{pmatrix}
\]

Each 0 and each 1 lies in a red row or column without *
Complexity of $M$-partition for chordal graphs

**Chordal graphs**
Is every $M$-partition problem polynomial?

**Crossed matrices**
If $M$ is crossed then the $M$-partition problem is polynomial
Chordal graphs

Is every $M$-partition problem polynomial?

Crossed matrices

If $M$ is crossed then the $M$-partition problem is polynomial

But

There exist non-crossed matrices $M$ for which the $M$-partition problem is NP-complete.

Feder-Hell-Klein-Nogueira-Protti 2005
Three Possibilities

The $M$-partition problem

- The $M$-partition problem is NP-complete.
- It can be solved in polynomial time, but with infinitely many minimal obstructions.
- There exists a finite set of minimal obstructions.

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
The $M$-partition problem

- NP-complete
Three Possibilities

The $M$-partition problem

- NP-complete

\[
\begin{pmatrix}
0 & * & *
\end{pmatrix}
\begin{pmatrix}
* & 0 & *
\end{pmatrix}
\begin{pmatrix}
* & * & 0
\end{pmatrix}
\]

- polynomial time solvable but with infinitely many minimal obstructions $(0^*0^*)$

- with a finite set of minimal obstructions $(0^*0^10^*)$
Three Possibilities

The $M$-partition problem

- NP-complete $\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$
- polynomial time solvable but with infinitely many minimal obstructions

Pavol Hell, Simon Fraser University
Partitions of Chordal Graphs
Three Possibilities

The $M$-partition problem

- NP-complete
  \[
  \begin{pmatrix}
  0 & * & * \\
  * & 0 & * \\
  * & * & 0
  \end{pmatrix}
  \]

- Polynomial time solvable but with infinitely many minimal obstructions
  \[
  \begin{pmatrix}
  0 & * \\
  * & 0
  \end{pmatrix}
  \]
Three Possibilities

The $M$-partition problem

- NP-complete
  \[
  \begin{pmatrix}
  0 & * & * \\
  * & 0 & * \\
  * & * & 0 \\
  \end{pmatrix}
  \]

- polynomial time solvable but with infinitely many minimal obstructions
  \[
  \begin{pmatrix}
  0 & * \\
  * & 0 \\
  \end{pmatrix}
  \]

- with a finite set of minimal obstructions

Pavol Hell, Simon Fraser University

Partitions of Chordal Graphs
Three Possibilities

The $M$-partition problem

- NP-complete \(
\begin{pmatrix}
0 & * & * \\
* & 0 & * \\
* & * & 0 \\
\end{pmatrix}
\)

- polynomial time solvable but with infinitely many minimal obstructions \(
\begin{pmatrix}
0 & * \\
* & 0 \\
\end{pmatrix}
\)

- with a finite set of minimal obstructions \(
\begin{pmatrix}
0 & * \\
* & 1 \\
\end{pmatrix}
\)
The $M$-partition problem

- NP-complete $\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$
- polynomial time solvable but with infinitely many minimal obstructions $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$
- with a finite set of minimal obstructions $\begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}$
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

Finite obstruction set

NP–c
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

Unfriendly matrices $M$

If $M$ contains \( \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \) or \( \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix} \), then the $M$-partition problem has infinitely many minimal obstructions.
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

Unfriendly matrices $M$

If $M$ contains $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix}$, then the $M$-partition problem has infinitely many minimal obstructions

Feder-Hell-Xie 2007
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

Friendly matrices $M$ of size at most 5

If $M$ does not contain \(
\begin{pmatrix} 0 & \ast \\ \ast & 0 \end{pmatrix}
\) or \(
\begin{pmatrix} 1 & \ast \\ \ast & 1 \end{pmatrix}
\), then the $M$-partition problem has finitely many minimal obstructions

Feder-Hell-Xie 2007
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

BUT

Friendly matrices $M$

There is a friendly matrix $M$ of size 6 such that the $M$-partition problem has infinitely many minimal obstructions.
The $M$-partition problem

For which matrices $M$ does it have a finite set of minimal obstructions?

BUT

Friendly matrices $M$

There is a friendly matrix $M$ of size 6 such that the $M$-partition problem has infinitely many minimal obstructions.

There is even a friendly matrix $M$ of size 9 such that the $M$-partition problem is NP-complete.

Feder-Hell-Xie 2007
If $M$ has no 1’s

A perfect graph $G$ has an $M$-partition $\iff$ it contains no induced $K_{c(M)+1}$
If $M$ has no 1’s

A perfect graph $G$ has an $M$-partition $\iff$ it contains no induced $K_{c(M)+1}$

$c(M)$ is the maximum clique size of the graph with “adjacency matrix” $M$
If $M$ has no 1’s

A perfect graph $G$ has an $M$-partition $\iff$ it contains no induced $K_{c(M)+1}$

c($M$) is the maximum clique size of the graph with “adjacency matrix” $M$

Similarly for matrices without 0
Normal matrices

\[
\begin{pmatrix}
0 & a_1 & a_2 & c_1 & c_2 \\
a_1 & 0 & a_3 & c_3 & c_4 \\
a_2 & a_3 & 0 & c_5 & c_6 \\
c_1 & c_3 & c_5 & 1 & b_1 \\
c_2 & c_4 & c_6 & b_1 & 1
\end{pmatrix}
\]

Either all or none of \(a_1, a_2, a_3\) are \(\ast\).

Either all or none of \(c_1, c_2, \ldots, c_6\) are \(\ast\).

Etc for the \(b_i\) (A normal matrix is crossed.)
Normal matrices

\[
\begin{pmatrix}
0 & a_1 & a_2 & c_1 & c_2 \\
a_1 & 0 & a_3 & c_3 & c_4 \\
a_2 & a_3 & 0 & c_5 & c_6 \\
c_1 & c_3 & c_5 & 1 & b_1 \\
c_2 & c_4 & c_6 & b_1 & 1
\end{pmatrix}
\]

Either all or none of \(a_1, a_2, a_3\) are \(*\)
Either all or none of \(c_1, c_2, \ldots, c_6\) are \(*\)
Etc for the \(b_i\)
### Normal matrices

$$
\begin{pmatrix}
0 & a_1 & a_2 & c_1 & c_2 \\
a_1 & 0 & a_3 & c_3 & c_4 \\
a_2 & a_3 & 0 & c_5 & c_6 \\
c_1 & c_3 & c_5 & 1 & b_1 \\
c_2 & c_4 & c_6 & b_1 & 1 \\
\end{pmatrix}
$$

Either all or none of $a_1, a_2, a_3$ are $\ast$

Either all or none of $c_1, c_2, \ldots, c_6$ are $\ast$

Etc for the $b_i$

(A normal matrix is crossed.)
Normal matrices

\[
\begin{pmatrix}
0 & a_1 & a_2 & c_1 & c_2 \\
 a_1 & 0 & a_3 & c_3 & c_4 \\
 a_2 & a_3 & 0 & c_5 & c_6 \\
 c_1 & c_3 & c_5 & 1 & b_1 \\
 c_2 & c_4 & c_6 & b_1 & 1 \\
\end{pmatrix}
\]

Feder-Hell 2006

If $M$ is normal, then the $M$-partition problem has finitely many perfect minimal obstructions
Every $M$-partition problem has finitely many minimal obstructions. 

Damaschke, 2004

Feder, Hell, Hochstättler, 2006
Cographs

Every $M$-partition problem has finitely many minimal obstruct’s
Every $M$-partition problem has finitely many minimal obstructions.

- Damaschke 2004
- Feder-Hell-Hochstättler 2006
Chordal graphs

A chordal graph $G$ can be partitioned into $\ell$ independent sets and $m$ cliques if and only if it does not contain an induced $(m+1)K_\ell + 1$.
A chordal graph $G$ can be partitioned into $\ell$ independent sets and $m$ cliques if and only if it does not contain an induced \((m+1)K_{\ell+1}\).
A chordal graph $G$ can be partitioned into $\ell$ independent sets and $m$ cliques if and only if it does not contain an induced $(m + 1)K_{\ell+1}$. 

**Example**

$$
\begin{pmatrix}
0 & * & * & * & * \\
* & 0 & * & * & * \\
* & * & 0 & * & * \\
* & * & * & 1 & * \\
* & * & * & * & 1 \\
\end{pmatrix}
$$
Chordal graphs

Which matrices $M$ yield $M$-partition problems with finitely many chordal minimal obstructions?
Small matrices with zero diagonal

\[ M = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} \]
Chordal graphs

Small matrices with zero diagonal

\[ M = \begin{pmatrix} 0 & a & * \\ a & 0 & c \\ * & c & 0 \end{pmatrix} \]
Chordal graphs

Small matrices with zero diagonal

\[ M = \begin{pmatrix} 0 & * & * \\ * & 0 & c \\ * & c & 0 \end{pmatrix} \]
Chordal graphs

Small matrices with zero diagonal

$$M = \begin{pmatrix} 0 & \ast & \ast \\ \ast & 0 & 0 \\ \ast & 0 & 0 \end{pmatrix}$$
Small matrices with zero diagonal

\[ M = \begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{pmatrix} \]
Small matrices with zero diagonal

\[ M = \begin{pmatrix} 0 & 1 & \ast \\ 1 & 0 & 1 \\ \ast & 1 & 0 \end{pmatrix} \]
Chordal graphs

Interesting small matrices

\[
\begin{pmatrix}
0 & * & *
\end{pmatrix}
\begin{pmatrix}
0 & 1 & *
\end{pmatrix}
\begin{pmatrix}
1 & 0 & *
\end{pmatrix}
\begin{pmatrix}
1 & 0 & *
\end{pmatrix}
\begin{pmatrix}
* & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & *
\end{pmatrix}
\begin{pmatrix}
* & 1 & 0
\end{pmatrix}
\begin{pmatrix}
* & 1 & 0
\end{pmatrix}
\begin{pmatrix}
* & 0 & 1
\end{pmatrix}
\begin{pmatrix}
* & 0 & 1
\end{pmatrix}
\begin{pmatrix}
* & * & 1
\end{pmatrix}
\begin{pmatrix}
* & * & 1
\end{pmatrix}
\]

Eight interesting theorems
Small matrices

For matrices of size smaller than 4, only

\[
M = \begin{pmatrix}
0 & * & * \\
* & 0 & 1 \\
* & 1 & 0
\end{pmatrix}
\text{ and } \begin{pmatrix}
0 & * & * \\
* & 0 & 1 \\
* & 1 & 1
\end{pmatrix}
\]

have an $M$-partition problem with infinitely many chordal minimal obstructions

Feder - Hell - Nekooei-Rizi 2011
Small matrices

For

\[ M = \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & 1 \\ * & 1 & 0 \end{pmatrix} \]

the \( M \)-partition problem has exactly three chordal minimal obstructions

Feder - Hell - Nekooei-Rizi 2011
Small matrices

For

\[
M = \begin{pmatrix}
0 & 1 & *\\
1 & 0 & 1 \\
* & 1 & 0 \\
\end{pmatrix}
\]

the $M$-partition problem has exactly three chordal minimal obstructions

Feder - Hell - Nekooei-Rizi 2011
Small matrices

For

\[
M = \begin{pmatrix}
1 & 0 & * \\
0 & 1 & * \\
* & * & 1
\end{pmatrix}
\]

the $M$-partition problem has finitely many chordal minimal obstructions

Feder - Hell - Nekooei-Rizi 2011
For

\[ M = \begin{pmatrix} 1 & 0 & \ast \\ 0 & 1 & \ast \\ \ast & \ast & 1 \end{pmatrix} \]

the \( M \)-partition problem has finitely many chordal minimal obstructions

Feder - Hell - Nekooei-Rizi 2011
Chordal graphs

What's next?

Which matrices $M$ lead to finitely many chordal minimal obstructions? Are they more numerous than those that lead to infinitely many?

Finitely many matrices with only two kinds of entries (Feder-Hell 2008)
What’s next?

Which matrices $M$ lead to finitely many chordal minimal obstructions?
What’s next?

- Which matrices $M$ lead to finitely many chordal minimal obstructions?
- Are they more numerous than those that lead to infinitely many?
What's next?

- Which matrices $M$ lead to finitely many chordal minimal obstructions?
- Are they more numerous than those that lead to infinitely many?

Finitely many
Chordal graphs

What’s next?

- Which matrices $M$ lead to finitely many chordal minimal obstructions?
- Are they more numerous than those that lead to infinitely many?

Finitely many

Matrices with only two kinds of entries (Feder-Hell 2008)
Small matrices

\[
M = \begin{pmatrix}
0 & * & * \\
* & 0 & 1 \\
* & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
M = \begin{pmatrix}
0 & * & * \\
* & 0 & 1 \\
* & 1 & 1
\end{pmatrix}
\]

have an \( M \)-partition problem with infinitely many chordal minimal obstructions.
Chordal graphs

Infinitely many

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Partitions of Chordal Graphs
$k$ diagonal 0’s and $k$ diagonal 1’s

Of the total $T$, more than $\sqrt{T}$ have infinitely many chordal minimal obstructions.
Diagonal of 1’s

Always finitely many chordal minimal obstructions?

(Or at most one 0)