Digraph minors and a WQO for tournaments

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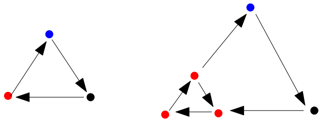
Digraph Minors?

Contracting an edge is not a good idea!

Can an acyclic digraph “contain” a directed cycle??

Digraph Minors

- A digraph G is a **Minor** of a digraph H if G can be obtained from a sub-digraph of H by identifying **strongly connected** subgraphs.



Robertson-Seymour theorem for Digraphs?

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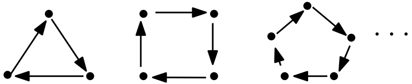

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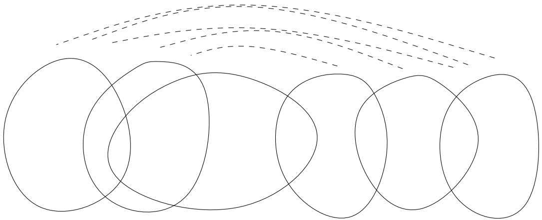
- For an infinite sequence of tournaments $G_1, \ldots$, do there exist i and j with $i < j$ such that G_i is a minor of G_j?
- The answer is "YES".
- (K, Seymour '11) : The class of all finite tournaments is a WQO under minor.

Path-decomposition for (undirected) graphs

For a graph G, we say a sequence $W_1, \ldots, W_r$ of subsets of $V(G)$ is a *path-decomposition* of G if this sequence satisfies the following conditions.

- $\bigcup_{i=1}^{r} W_i = V(G)$,
- for $1 \leq h < i < j \leq r$, $W_h \cap W_j \subseteq W_i$, and
- if $uv \in E(G)$, then $u, v \in W_h$ for some h.



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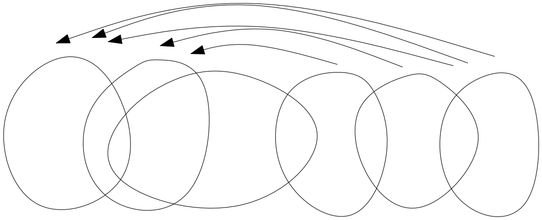
- if $uv \in E(G)$, then $u \in W_h$ and $v \in W_j$ for some $h \geq j$.



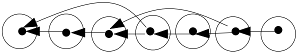
Path-width for digraphs

For a path-decomposition $W_1, \ldots, W_r$ of G, *path-width* of the path-decomposition is defined as

- $\max_{1 \leq i \leq r} |W_i| - 1$

The *path-width* of G is defined as the smallest path-width of a path-decomposition of G.

Example : An acyclic digraph has path-width 0.



Path-width for digraphs

(K, Seymour '11) If a digraph G has path-width $\leq k$, then every minor of G has path-width $\leq k$ as well.

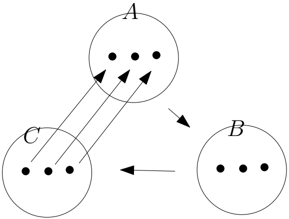
Bounded path-width

An undirected graph has “big” path-width if and only if it has “big” binary tree as a minor.

What are obstructions for the class of digraphs with bounded path-width?

k-triple

A tournament G is called a k-triple if $V(G) = A \cup B \cup C$ where $A = \{a_1, \ldots, a_k\}, B = \{b_1, \ldots, b_k\}$, and $C = \{c_1, \ldots, c_k\}$ and A is complete to B and B is complete to C and $c_i a_i \in E(G)$ for each $i = 1, \ldots, k$.



Bounded path-width

G has "big" path-width if and only if G has "big" k-triple.

(Fradkin, Seymour '10) For every set $\mathcal{S}$ of tournaments, the following are equivalent:

- Every member of $\mathcal{S}$ has path-width at most k for some k.
- Every member of $\mathcal{S}$ does not have k-triple for some k.

Bounded path-width

(K, Seymour '11) For every set $\mathcal{S}$ of tournaments, the following are equivalent:

- Every member of $\mathcal{S}$ has path-width at most k for some k.
- Every member of $\mathcal{S}$ does not contain H as a minor for some digraph H.

Goal

1) The class of all tournaments is a WQO under minor.

2) For each k, the class of all tournaments with path-width at most k is a WQO under minor.

Proof that 2) implies 1).

- Let $T_1, \ldots$ be an infinite sequence of tournaments.

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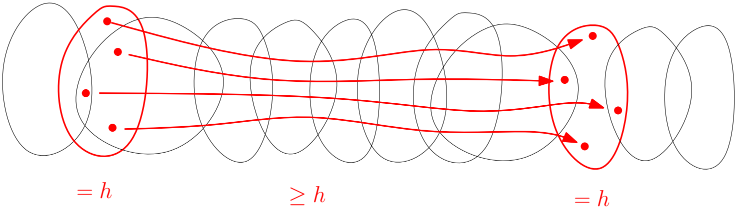
- We may assume T_i does not contain T_1 as a minor for each $i = 2, \ldots$.
- Then every member of $\{T_2, \ldots\}$ has path-width at most k for some k.
- From 1), there exists $2 \leq i < j$ such that T_i is a minor of T_j.

Liked path-decomposition

Let $W_1, \ldots, W_r$ be a path-decomposition of T. We say the path-decomposition is linked if $|W_h| = |W_j| = h$ and $|W_i| \leq h$ for some i, j, k with $i < j < k$, then there exist h vertex disjoint paths from W_h to W_j.



Liked path-decomposition

(K, Seymour '11) : Let G be a tournament with path-width k. Then there exists a **linked path-decomposition** $W_1, \ldots, W_r$ with path-width k.

Idea : Say a path-decomposition has $(x_0, \ldots, x_{k+1})$ - type where x_i is the number of bags of size i. Give a partial order of types as $(x_0, \ldots, x_{k+1}) \lesssim (y_0, \ldots, y_{k+1})$ iff there exists some t such that $x_i = y_i$ for every $i \leq t$ and $x_t \leq y_t$. Then, the path-decomposition with "maximal" type is linked.

Liked path-decomposition

$\mathcal{G}_{k,h}$ denotes the class of all tournaments G satisfies the following.

- There exists a **linked path-decomposition** $W_1, \ldots, W_r$ of G such that $h \leq W_i \leq k+1$ for every i and $|W_1| = |W_r| = h$.

Remark : If G has path-width at most k, then $G \in \mathcal{G}_{k,0}$.

Goal

1) For each k and h, $\mathcal{G}_{k,h}$ is a WQO under minor.

2) For each k, the class of all tournaments with path-width at most k is a WQO under minor.

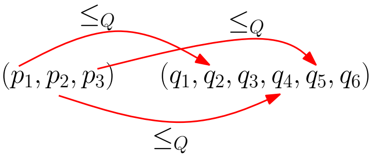
3) The class of all tournaments is a WQO under minor.

Strategy : Prove 1) by induction on $k - h$.

Higman's Theorem

(Higman '52) If $(Q, \leq_Q)$ is a wqo, then so is $(Q^{<\omega}, \leq_{Q^{<\omega}})$.

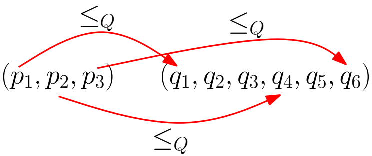
$(p_1, \ldots, p_m) \leq_{Q^{<\omega}} (q_1, \ldots, q_n)$ iff
$m \leq n$ and there exists $1 \leq i_1 < \ldots < i_m \leq n$ such that $p_j \leq_Q q_{i_j}$
for each $j = 1, \ldots, m$.



(Modified) Higman's Theorem

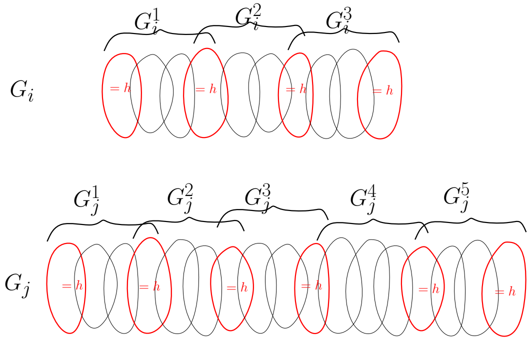
If $(Q, \leq_Q)$ is a wqo, then so is $(Q^{<\omega}, \leq_{Q^{<\omega}})$.

$(p_1, \ldots, p_m) \leq_{Q^{<\omega}} (q_1, \ldots, q_n)$ iff
$m \leq n$ and there exists $1 = i_1 \leq i_2 < \ldots < i_{m-1} < i_m = n$ such that $p_j \leq_Q q_{i_j}$ for each $j = 1, \ldots, m$.



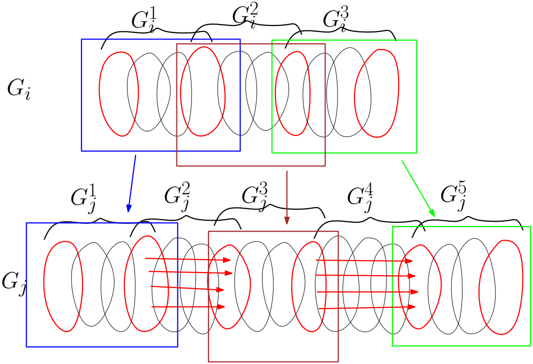
Proof Sketch

Induction on k. For fixed k, we apply induction on $k+1-h$.
Base Case : $\mathcal{G}_{k,h}$ is a finite set if $k+1-h=0$.
Take G_i and G_j from $\mathcal{G}_{k,h}$.



Proof Sketch

Roughly, the small pieces are well-quasi-ordered.
We apply the modified Higman's Theorem.



Proof Sketch

Problem 1 : If one of the h vertex-disjoint paths has length 1?

We need vertices to be labeled.

Problem 2 : What if the order of the h vertices are different?

We need those special $2h$ vertices to be rooted.

Conclusion and Future Problems

The class of all semi-complete digraphs is a well-quasi-order under minor.(not true for super-tournaments)

What about for digraphs with bounded stability number?

What about for Labeled Minor / Rooted Minor?

Thank you!