Lower Bounds on the Complexity of MSO_1 Model-Checking

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Algorithmic Meta Theorems

Theorems that identify *classes* of tractable problems, rather than a few isolated problems.

Examples

- All graph properties expressible in MSO_2 can be decided in linear time on graphs of bounded treewidth [Courcelle, 1990].
- All problems in MAX SNP have constant-factor approximation algorithms [Papadimitriou and Yannakakis, 1991].
- Compact parameterized problems expressible in CMSO admit polynomial kernels on graphs of bounded genus [Bodlaender et al, 2010].

Uses

- Quick way of checking whether a problem admits an algorithm of a particular kind.

Courcelle's Theorem

Theorem (Courcelle, 1990)

Any graph property definable in MSO_2 *can be decided in linear time on any class of graphs of bounded treewidth.*

MSO_2 – monadic second-order logic with quantification over *sets* of vertices and/or edges

Expressible in MSO_2: HAMILTONIANCYCLE, VERTEXCOVER, ...

3-COLOURABILITY:

$$\exists V_1, V_2, V_3 \quad \Big[\forall v\,(v \in V_1 \vee v \in V_2 \vee v \in V_3) \wedge \bigwedge_{i=1,2,3} \forall v, w\,(v \notin V_i \vee w \notin V_i \vee \neg\, adj(v, w))\Big]$$

Courcelle's Theorem – Lower Bounds

The theorem may also be stated as:

Theorem (Courcelle, 1990)

Let $\mathcal{C}$ be any class of graphs of bounded tree-width. Then $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$ is decidable in linear time.

$\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$ – the MSO_2 model-checking problem on $\mathcal{C}$:
Given a $G \in \mathcal{C}$ and $\varphi \in \mathrm{MSO}_2$ check whether $G \models \varphi$.

Questions:

- Are there classes of graphs of unbounded treewidth such that Courcelle's Theorem still holds?
  YES! (in XP) [Makowsky and Mariño, 2004]
- How fast must the treewidth grow for Courcelle's theorem to fail?
  Poly-logarithmically [Kreutzer and Tazari, 2010]

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Unbounding Tree-Width

Definition (Kreutzer and Tazari, 2010)

The treewidth of a graph class $\mathcal{C}$ is *strongly unbounded by $f : \mathbb{N} \to \mathbb{N}$* if for all $n \in \mathbb{N}$ there exists $G_n \in \mathcal{C}$ with

- $f(|G_n|) \leq tw\ G_n$ *unbounded*
- $n \leq tw\ G_n \leq n^\gamma$, for some fixed γ *dense*
- G_n can be constructed in time 2^{n^ϵ}, for some fixed $\epsilon < 1$ *constructable*

strongly unbounded poly-logarithmically:

$\log^c(|G_n|) \leq tw\ G_n$ for all $c \geq 1$

Recent known results

Theorem (Kreutzer and Tazari, SODA'10)

Let $\mathcal{C}$ be a graph class with the following properties:

1. *the treewidth of $\mathcal{C}$ is strongly unbounded poly-logarithmically*
2. *$\mathcal{C}$ is closed under Γ-colourings*

Then $\mathrm{MC}(\mathrm{MSO}_2\text{-}\Gamma, \mathcal{C})$ is not in XP ($|G|^{f(|\varphi|)}$ for any computable f), unless the Exponential-Time Hypothesis (ETH) fails.

Theorem (Kreutzer and Tazari, LICS'10)

Let $\mathcal{C}$ be a graph class with the following properties:

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Our Results I

Theorem

Assume a (suitable but fixed) finite label set L. Let $\mathcal{C}$ be a graph class with the following properties:

1. *the tree-width of $\mathcal{C}$ is densely unbounded poly-logarithmically*
2. *$\mathcal{C}$ is closed under taking subgraphs*

Then $\mathrm{MC}(\mathrm{MSO}_1\text{-}L, \mathcal{C}^L)$ is not *in XP unless the nonuniform Exponential-Time Hypothesis (nonuniform ETH) fails.*

MSO_1 – no quantification over sets of edges

MSO_1-L – extension of MSO_1 with vertex-label predicates for a finite set of labels L

$\mathcal{C}^L$ – the class of all L-vertex labelled graphs from $\mathcal{C}$

nonuniform ETH – SAT not in $2^{o(n)}$ with subexponential advice

Main Improvements

1. Our definition of "densely unbounded" avoids the constructability requirement in the definition of "strongly unbounded".

   We "pay" for this by using a stronger complexity assumption: *Nonuniform ETH*.

2. *Kreutzer and Tazari*: MSO_2 on unlabelled graphs
   Our result: MSO_1-*L*.

3. Much simpler, streamlined proof.

MSO_2 vs MSO_1-L

MSO_1-L is much weaker then MSO_2:

- HAMILTONIANPATH cannot be expressed in MSO_1-L
- Extending [Courcelle, Makowski, Rotics, 2000] from MSO_1 to MSO_2 would mean EXP=NEXP

Labels do not matter:

- Many results concerning MSO_1 can be formulated with or without labels.
- Both Courcelle's theorem and [CMR00] can be extended with labels.

Proof Outline

- At a high level, our proof technique is similar to Kreutzer and Tazari: a *multi-step reduction from* SAT.
- Proof *shorter* mainly because we do not need to tediously construct a “skeleton” in the class $\mathcal{C}$ suitable for reduction. It comes “for free” from the *oracle advice function* which comes with the nonuniform computing model.
- We *avoid* the need for MSO_2 by using *strong edge colourings* to simulate certain edge-sets inside MSO_1-*L*.

High-level Proof Description

Reduce SAT to $\mathrm{MC}(\mathrm{MSO}_2, \mathcal{C})$.

- *Input:* A SAT formula F of length n.
- *Question:* Is F satisfiable?

Reduction

1. Construct $G_n \in \mathcal{C}$ of treewidth n^d s.t. $\log^c(|G_n|) < tw\, G_n$ and $c > d$.
   - Strongly poly-logarithmically unbounded.
2. Encode F in a subgraph of G_n.
   - Using closure under subgraphs.
3. Define an MSO-formula φ (independent of F) s.t. F satisfiable iff $G_n \models \varphi$.
   - Deciding $G_n \models \varphi$ takes time $2^{n^{c/d} \cdot f(|\varphi|)}$, subexponential in $|F|$.

Tools Used I

Grid-like graphs (minors)

A pair $(G, \mathcal{P})$ such that:

1. G is the union of all the paths in $\mathcal{P}$,
2. each path in $\mathcal{P}$ has at least two vertices, and
3. the *intersection graph* $I(\mathcal{P})$ of the path collection is bipartite.

Theorem (Reed and Wood, 2008)

Every graph with tree-width at least $c\ell^4\sqrt{\log \ell}$ contains a subgraph which is grid-like of order ℓ, for some constant c.

Order of a grid-like graph: the maximum integer ℓ such that the intersection graph $I(\mathcal{P})$ contains a K_ℓ-minor

Tools Used II

Strong Edge Colourings

An assignment of colours to the edges of a graph such that no path of length three contains the same colour twice.

Theorem (Cranston, 2006)

Every graph of maximum degree 4 *has a strong edge-colouring using at most* 22 *colours. This colouring can be found with a polynomial-time algorithm.*

Tools Used III

MSO_1 interpretation in {1,3}-regular graphs

Theorem (Ganian et al, 2010)

The MSO_1 *theory of all simple graphs has an efficient interpretation in the* MSO_1 *theory of all simple* $\{1,3\}$*-regular graphs. Furthermore, this efficient interpretation I can be chosen such that, for every* MSO_1 *formula* ψ*, the resulting property* ψ^I *is invariant under subdivisions of edges.*

Our Results II

Our main theorem can be strengthened by taking a stricter assumption:

Theorem

Let $\mathcal{C}$ be a graph class with the following properties:

1. *the tree-width of $\mathcal{C}$ is densely unbounded poly-logarithmically*
2. *$\mathcal{C}$ is closed under taking subgraphs*

Then $\mathrm{MC}(\mathrm{MSO}_1\text{-}L, \mathcal{C}^L)$
for every finite set of labels L such that $|L| = (|\varphi|)$
is not *in* XP *unless*
$\mathrm{PH} \subseteq \mathrm{DTIME}(2^{o(n)})/\mathrm{SubEXP}$.

Our Results III

This is an extension of *[Ganian et al, 2010]*

Theorem

Let L be a finite set of labels, $|L| \geq 47$. Unless the nonuniform Exponential-Time Hypothesis fails, there exists no directed width measure δ satisfying following three properties:

1. *δ is monotone under taking subdigraphs;*
2. *δ largely surpasses the tree-width of underlying undirected graphs; and*
3. *for all L-vertex-labelled digraphs D and all formulas $\varphi \in \mathrm{MSO}_1$-L, the problem of deciding whether $D \models \varphi$ is solvable in time $O(|D|^{f(\delta(D),|\varphi|)})$ for some computable f.*

And Now for Something Completely Different . . .

Food!