On the constructive power of monadic second-order logic

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References : B.C. & J. Engelfriet, Graph structure and monadic second-order logic, book to be published by *Cambridge University Press* (April 2012)

B.C. & A. Blumensath, Monadic second-order graph orderings, *in preparation*.

See for both : http://www.labri.fr/perso/courcell/ActSci.html

Presentation of the talk

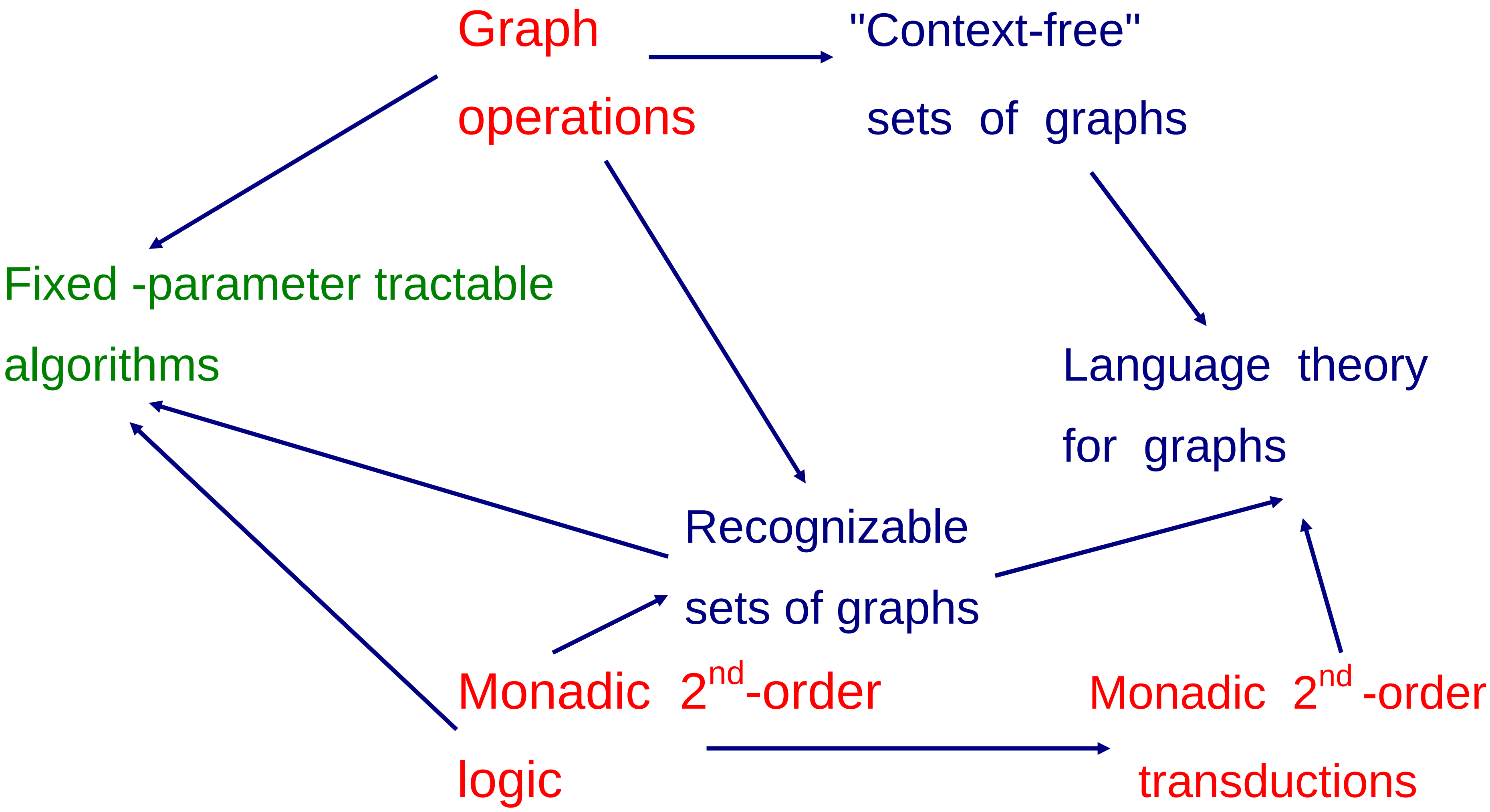
Monadic Second-Order (MSO) logic can express graph properties and mappings from (labelled) graphs to (labelled) graphs.

Main facts :

1. MSO graph properties are FPT with respect to clique-width and tree-width.
2. So are MSO counting and optimizing functions.

3. MSO definable sets of graphs are *recognizable* (by finite congruences, there is no good notion of graph automaton).
4. Recognizable sets behave well with respect to the *equational* sets (which generalize context-free languages).
5. There is no good notion of automaton-based transducer. But the *MSO definable transductions* behave well w.r.t. equational and recognizable sets (respectively : direct and inverse preservation results).

An overview chart



Summary

1) MSO logic without and with edge quantifications.
2) MSO definitions of orientations.
3) MSO definitions of linear orders.
4) Other constructions.
5) Open problems.

Two types of MSO formulas or rather two logical graph representations

MSO formulas

MSO_2 formulas with edge quantifications = MSO formulas over incidence graphs

$G = (V_G , edg_G(.,.))$

$Inc(G) = (V_G \cup E_G, inc_G(.,.))$

for G undirected : $inc_G(e,v) \Leftrightarrow$ v is a vertex (in V_G) of edge e (in E_G)

FPT for clique-width

FPT for tree-width

For G directed : $Inc(G) = (V_G \cup E_G, inc1_G(.,.), inc2_G(.,.))$ (1=tail, 2=head)

Typical MSO graph properties

MSO properties : *3-colorablility*

$\exists X,Y$ ("X,Y are disjoint" $\wedge$ $\forall u,v$ { edg(u,v) $\Rightarrow$

[$(u \in X \Rightarrow v \notin X) \wedge (u \in Y \Rightarrow v \notin Y) \wedge (u \notin X \cup Y \Rightarrow v \in X \cup Y$] })

Connectedness, negation of :

$\exists Z\, (\exists x \in Z \wedge \exists y \notin Z \wedge \forall u,v\, (u \in Z \wedge edg(u,v) \Rightarrow v \in Z)\,)$

Planarity (via two forbidden minors K_5 and $K_{3,3}$)

Perfectness (via forbidden holes and anti-holes)

Typical MSO_2 graph properties

MSO_2 property that is *not MSO* :

has a *perfect matching* or

has a *Hamiltonian circuit* or

has a *spanning tree of degree* ≤ 3

The expressions have the form:

"There exists a set of edges that is …"

Monadic Second-Order definitions of orientations

Particular monadic second-order transductions:

G undirected → G', orientation of G

Two cases : by MSO or by MSO_2 formulas

All cases : with parameters (that "guess" an appropriate coloring or spanning tree or …)

By MSO_2 formulas (the easiest case, of course)

Idea: for a graph G, "guess", by means of 2 parameters (X, set of edges and Y, set of vertices), a *depth-first* rooted spanning forest F ("depth-first" - or "normal" - means that every edge of the graph links a vertex and one of its ancestors w.r.t. F).

From F one obtains an acyclic orientation of G.

An additional parameter Z can specify the set of edges "to be reversed".

Fact: By means of 3 parameters X,Y,Z (over Inc(G)), one can specify by MSO_2 formulas *all orientations* of a given graph.

Formally : there are MSO_2 formulas $\alpha(X,Y,Z)$ and $\beta(X,Y,Z,u,v)$ such that, for every graph G,

1) there exist X,Y,Z satisfying α in Inc(G) , meaning that X,Y define a depth-first rooted spanning forest and Z is a set of edges,

2) for every X,Y,Z satisfying α and every two adjacent vertices u,v:

$$\beta(X,Y,Z,u,v) \Rightarrow \neg \beta(X,Y,Z,v,u),$$

hence, $\beta(X,Y,Z, .,.)$ defines one orientation of each edge.

Furthermore, for every such X,Y and every orientation H of G, there is Z such that $\beta(X,Y,Z, .,.)$ defines the orientation H.

Consequences :

If Q is an MSO_2 property of directed graphs, then the property of undirected graphs G :

$$P(G) \Leftrightarrow G \text{ has an orientation satisfying } Q$$

is MSO_2. (False for MSO).

Analogy : Tree-width is invariant under changes of orientation. Here MSO_2 formulas can specify arbitrary (changes of) orientation(s).

By MSO formulas : more difficult

Fact: No pair of MSO formulas can specify *at least one orientation* of any graph.

Proof : Assume this possible with *p* parameters $X_1, \ldots, X_p$.

Take a clique K_n with $n > 2^p$. There are adjacent vertices u, v that belong to the same sets X_i, hence

$$\beta(X_1, \ldots, X_p, u, v) \Leftrightarrow \beta(X_1, \ldots, X_p, v, u),$$

so that the edge u—v is not oriented by $\beta(X_1, \ldots, X_p, ., .)$.

(There is an automorphism that preserves the sets $X_1, \ldots, X_p$.)

Hence, MSO orientability *needs* some combinatorial conditions.

Case 1 : Defining an orientation of p-colorable graphs.

Parameters $X_1, \ldots, X_p$ are intended to specify a p-coloring. The orientation is $u \rightarrow v$ if $u \in X_i$, $v \in X_j$ and $i < j$.

One defines particular orientations, not all of them.

Remarks : With 4 parameters (4 colors), one can define *some* orientations of each planar graph. With 80 parameters (80 colors, by Raspaud & Sopena), one can define *all* orientations of each simple planar graph. (Actually 2 and 7 parameters can encode 4 and 80 colors respectively).

Case 2 : Defining an orientation of indegree $\leq$ p.

Let m = $2^{2p(p+1)+1}$ -1. There is a tournament T with V_T = [m] such that for every oriented graph H of indegree $\leq$ p, there is a homomorphism h: H → T (by Nesetril et al.). Such h can be specified by parameters $X_1, \ldots, X_m$.

Then u → v ⇔ u ∈ X_i , v ∈ X_j and i → j in T.

Consequence : Uniform p-sparsity of G is MSO expressible.

Means : $|E_K| \leq p.|V_K|$ for every subgraph K of G. Because (by Nash-Williams) it is equivalent to the existence of an orientation of indegree $\leq$ p. (No MSO expression of the definition).

Monadic Second-Order definitions of *linear orders*

Facts : 1) A linear order yields an orientation.

2) Impossible to define all linear orders, even with edge quantifications.

Proof : *Counting argument* by considering P_n for large n.

(n! linears orders but only 2^{pn} different ones defined from p parameters).

3) Impossible to define with fixed MSO_2 formulas a linear order on each graph.

Proof : Assume this possible with p parameters and consider an edgeless graph with n vertices, $n > 2^p$. Same argument based on automorphisms as for orientations.

By MSO_2 formulas (the easiest case)

Case 1 : Rooted trees of degree ≤ *d*.

With *d* parameters (defining sets of edges), a formula can order the successors of every node. Another one can order lexicographically the access paths to nodes.

Case 2 : Graphs with a spanning tree of degree ≤ *d*.

One parameter can choose such a tree, and we use Case 1.

In particular: Cliques : *d* = 1;

3-connected planar graphs : *d* = 3, (by Barnette).

Now some necessary conditions.

Basic "separation" condition: If an MSO_2 formula β of quantifier-height *h* and using *p* parameters orders a graph G, this graph has $<$ f(*h*, *p*) connected components for some fixed function f.

Proof sketch : Let G with connected components $C_1, \ldots, C_N$, and chosen *p* parameters. Let u_i be a vertex of C_i.

Whether $\beta(X_1, \ldots, X_p, u_i, u_j)$ is true depends (by a fixed function) on $(\Theta_i, \Theta_j, \{ \Theta_k / k \neq i,j \})$ where Θ_i is the "*h*-theory" of u_i in C_i i.e., the *finite* set of formulas $\gamma(X_1, \ldots, X_p, w)$ of quantifier-height $\leq$ *h* true in C_i with u_i as value of w.

Again: Whether $\beta(X_1, \ldots, X_p, u_i, u_j)$ is true depends only on $(\Theta_i, \Theta_j, \{ \Theta_k / k \neq i,j \})$ where Θ_i is the "*h*-theory" of u_i in C_i.

If $N \geq$ some $f(h, p)$, there are u_i and u_j with same "*h*-theories" and thus $\beta(X_1, \ldots, X_p, u_i, u_j) \Rightarrow \beta(X_1, \ldots, X_p, u_j, u_i)$; u_i and u_j are not ordered. (We use logic, not only automorphisms).

Generalized necessary "separation" condition SEP*:* If a class of graphs is MSO_2 orderable, there is a function g on integers such that for every graph G and every set X of k vertices the graph G – X has < g(k) connected components. (Formally : $G \in Sep(g)$.)

This condition is necessary but not sufficient.

Counter example: the graphs $K_{n, 2^{2^n}}$

Proof:

Not MSO_2 orderable by easy argument using automorphisms.

Fact: If f is a strictly increasing function, then the graphs $K_{n, f(n)}$ are in Sep(f).

Question : Which additional condition makes it sufficient ?

Answer 1 : Excluding $K_{p,p}$ as a minor.

Remark: By using a different logic (First-order logic with least fixed-points over sets of k-tuples), M. Grohe can order the graphs of every class that excludes a minor and he gets a logic that captures PTIME on these classes (LICS 2010).

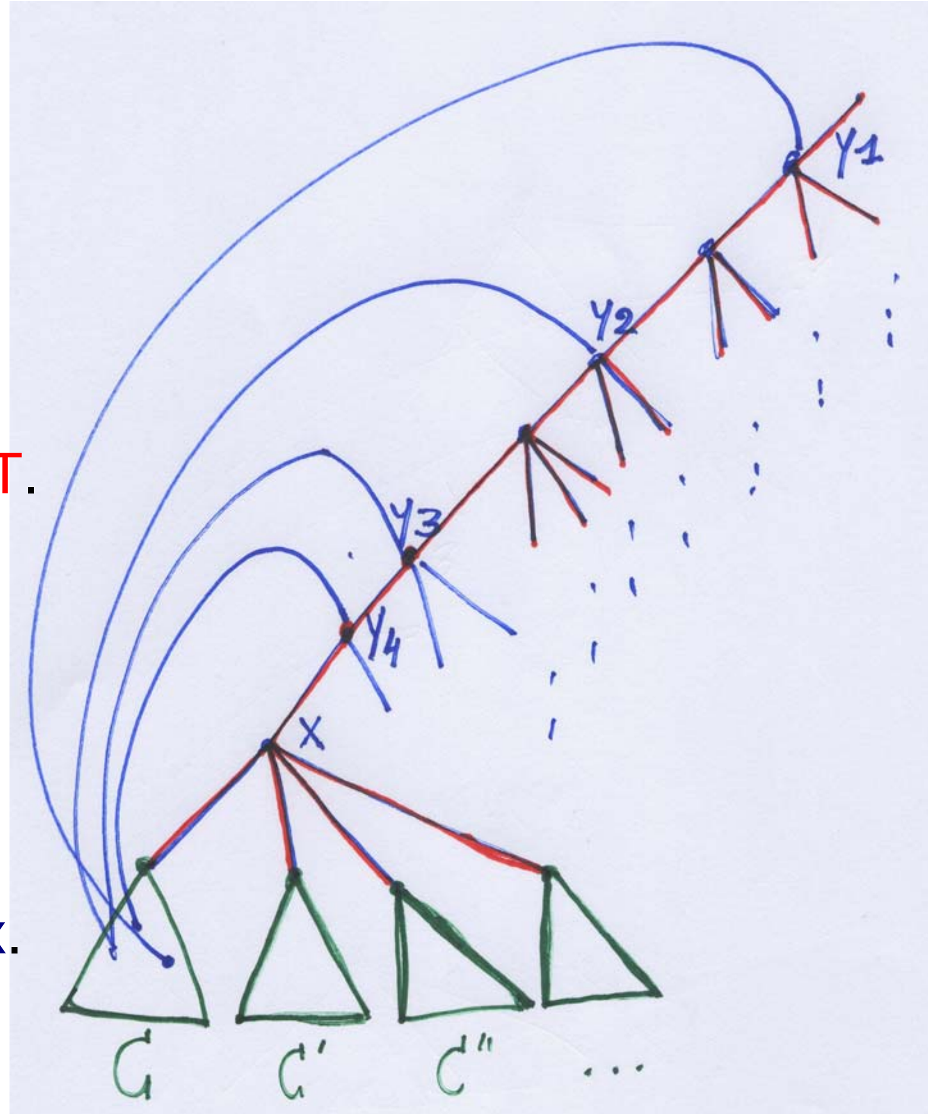
Proof sketch : Let G connected without $K_{p,p}$ as a minor satisfy SEP for some function g.

Let T be a depth-first spanning tree (chosen by some parameter).

We must order the successors of every node x of T.

If C is a "successor component" of x, let Anc(C) = (y_1, …, y_q).

1) We order lexicographically w.r.t. Anc(C) the successors of x.



We need to order two components with same ancestor list.

Let Anc(C) = ($y_1, \ldots, y_q$).

2) If $q \geq p$, there are less than p successor components of x with same ancestor list (otherwise $K_{p,p}$ is a minor of G); with p-1 parameters, one can order them (as for trees of bounded degree).

3) If $q < p$, there are < g(p+1) successor components of x with same list of ancestors (we use SEP by deleting x, $y_1, \ldots, y_q$) with g(p+1) parameters, one can order them.

Finally, we order lexicographically the access paths to the nodes of T (the vertices of G).

If G is not connected, it has < g(0) connected components.

Improvement :

If G has no minor $K_{p,p}$ and Sep(G,p) $\leq$ d , then Sep(G, k) $\leq (p+d).k.2^k$ for every $k \geq p$.

Hence, a class without $K_{p,p}$ as a minor is an MSO_2 –orderable if and only if Sep(G,p) is bounded for G in this class.

(The combinatorial condition need not consider Sep(G,k) for all values of k, but only k = p.)

Question : Can one replace "excluding $K_{p,p}$" by "r-sparse"?

No: Consider the incidence graphs of $K_{n,f(n)}$ with f not "elementary".

Answer 2, about dense graphs.

A set of complete bipartite graphs $K_{m,n}$ with $m \geq n$ is

MSO_2 –orderable $\Leftrightarrow$

it satisfies $m \leq a^n$ for some a $\Leftrightarrow$

it satisfies SEP($\lambda k.\ a^k$) for some a.

Hence, for cographs, SEP *does not imply* MSO_2–orderability.

Question : Find necessary and sufficient conditions for a set of cographs to be MSO_2 –orderable.

Answer 3, about *split* graphs (particular chordal graphs).

Similar fact:

A set of split graphs is MSO_2–orderable $\Leftrightarrow$

it satisfies SEP(λk. a^k) for some a.

Again : SEP *does not imply* MSO_2–orderability.

There exists an MSO_2–orderable set of *chordal graphs* that is not included in SEP(λk. a^k) for any a. (We build graphs G such that Sep(G,k) = k!).

Linear ordering by MSO formulas.

Observation : Cliques are MSO_2 –orderable but not MSO–orderable (for MSO, they are equivalent to edgeless graphs).

Hence, we need a stronger combinatorial condition than SEP.

SEP is based on vertex separators (cf. tree-decompositions).

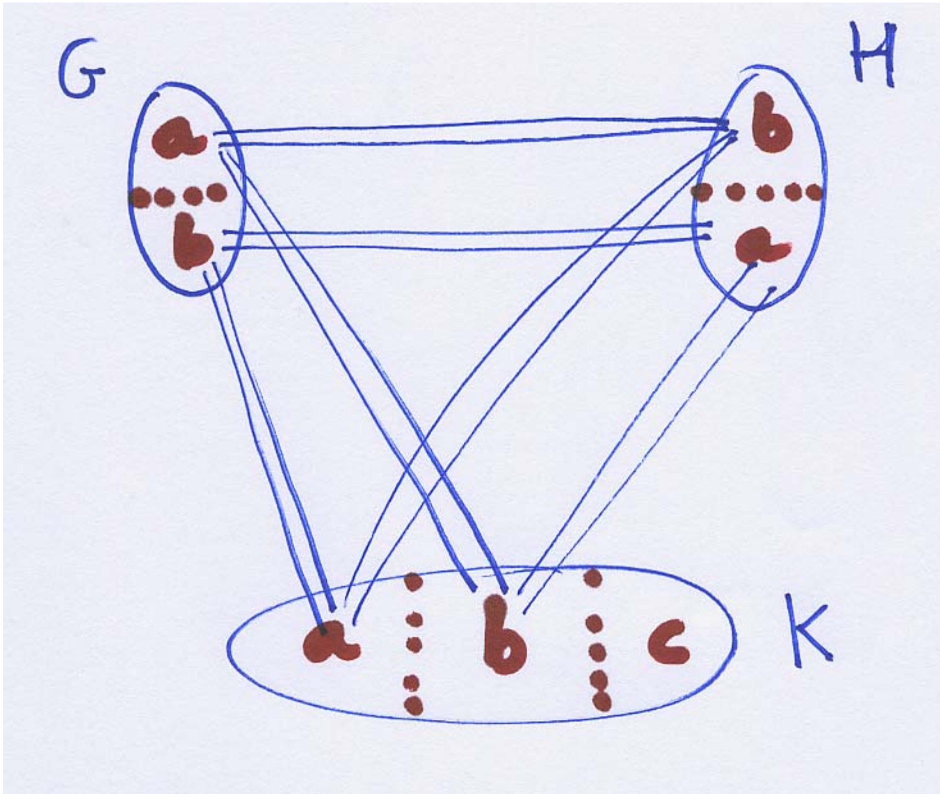
We will introduce certain edge-separations by complete bipartite graphs (cf. the definition of clique-width).

Definition 1 : A family of *associative and commutative* “clique-width” operations.

G, H simple, undirected graphs with vertex labels in [k],

R $\subseteq$ [k] X [k], symmetric,

G $\otimes_R$ H is G $\oplus$ H with edges between every vertex of G labelled by *a* and every vertex of H labelled by *b* such that (*a*,*b*) $\in$ R.



Example: G $\otimes_R$ H $\otimes_R$ K

R = {(a,b),(b,a)}

Definition 2 : Let G simple and undirected.

Cut(G,k) is the maximum number n of graphs $H_1, \ldots, H_n$ with labels in [k], such that (just another way to split graphs):

$$G = H_1 \otimes_R \ldots \otimes_R H_n \quad \text{for some } R \text{ symmetric} \subseteq [k] \times [k].$$

$G \in$ CUT(g) if Cut(G,k) $\leq$ g(k) for every k.

A necessary condition

Proposition : Let G be simple, undirected, MSO-ordered by a formula with *p* parameters and quantifier-height *h*.

Then Cut(G,k) $\leq$ f(k, *p*, *h*).

Proposition : CUT(g) $\subseteq$ SEP(g') where $g'(k) = g(k+2^k)$,

i.e. CUT $\Rightarrow$ SEP.

Conversely,

Proposition : If G is uniformly *q*-sparse :

G $\in$ SEP(f) $\Rightarrow$ G $\in$ CUT(f') where $f'(k) = g(6.k^2.q^2)$.

For graphs that are uniformly *q*-sparse :

SEP $\Leftrightarrow$ CUT,

bounded tree-width $\Leftrightarrow$ bounded clique-width,

MSO_2 is equivalent to MSO.

Proposition: A class C of cographs

is MSO-orderable

$\Leftrightarrow C \subseteq$ CUT(g) for some function g,

$\Leftrightarrow$ the modular decomposition trees of its cographs have bounded degree.

Cographs labelled by a are defined by terms over $\oplus$ and $\otimes$ ($= \otimes_{\{ (a,a) \}}$), handled as associative and commutative operations of variable arity; the terms representing their modular decompositions have no two consecutive $\oplus$ or $\otimes$ on any branch.

Question : What about a class C of bounded clique-width ?

What about chordal graphs ?

MSO-orderability of chordal graphs $\geq$

MSO-orderability of incidence graphs $\equiv$

MSO_2-orderability of all graphs.

CUT for incidence graphs $\equiv$ SEP for basic graphs.

No hope for bipartite graphs :

Because arbitrary graphs can be encoded as bipartite graphs.

The encoding preserves MSO-orderability and CUT.

Other MSO-definable constructions ; open questions

- Monadic second-order transductions (MST). They are more general than those presented here : the output structure may have a domain k times larger than that of the input structure. They still use parameters.

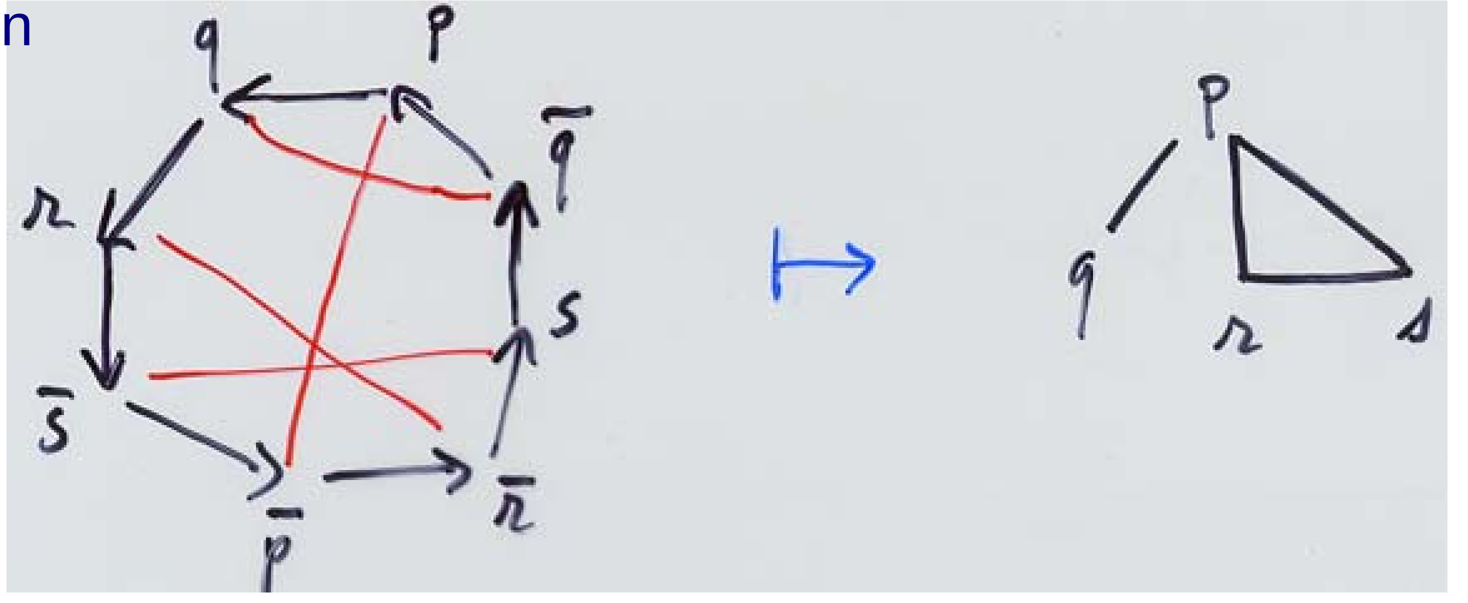
- A class of graphs C has bounded tree-width $\Leftrightarrow$

$$\mathrm{Inc}(C) \subseteq \mathrm{MST}(\mathrm{Trees}) \quad \text{(using } \mathrm{MSO}_2\text{)}$$

- A class of graphs C has bounded clique-width $\Leftrightarrow$

$$C \subseteq \mathrm{MST}(\mathrm{Trees}) \quad \text{(using MSO).}$$

- The mapping from a *linearly ordered* graph to its (unique) modular decomposition (or to its "split" decomposition, by Cunnigham) is an MST (not using edge quantifications).
- The mappings between ordered circle graphs and their chord intersection diagrams



It follows that a set of circle graphs has bounded clique-width if and only if their chord intersection diagrams have bounded tree-width.

- *Some* planar embedding of a *linearly ordered* connected planar graph can be defined by an MST. (A linear order is MSO-definable for 3-connected planar graphs; a planar embedding of a star is a circular order of its vertices of degree 1.)

- Some tree-decomposition of width k for any graph of tree-width $\leq k \leq 3$ (B.C, Kaller) or of path-width $\leq k$ (Kabanets).

- *Conjecture :* For each $k > 3$, there is an MST constructing a tree-decomposition of width $\leq f(k)$ of any graph of tree-width $\leq k$, where f is a fixed function.

It would yield, for any class of graphs of bounded tree-width, an equivalence between *recognizability* and *CMSO-definability* (i.e. definability by MSO formulas that can use set predicates meaning that the cardinality of a set is a multiple of a fixed integer).

Recognizability means here recognizability by finite automata on labelled trees encoding tree-decompositions.

It would also give for such classes the equivalence between *CMSO-definability* and *order-invariant MSO-definability* (i.e., MSO-definability with the help of an arbitrary linear order. Modulo-counting set predicates are order-invariant MSO-definable.

Other open questions

1) MSO and MSO_2 orderability of particular classes of graphs:

Which conditions in addition to CUT and SEP ?

2) Graphs omitting a fixed graph H as a minor have a particular tree-structure (defined by Robertson & Seymour).

Is this structure constructible by an MST ?

(Of course, one first need to prove the conjecture for graphs of bounded tree-width).