Obstructions to Linear Rankwidth 1

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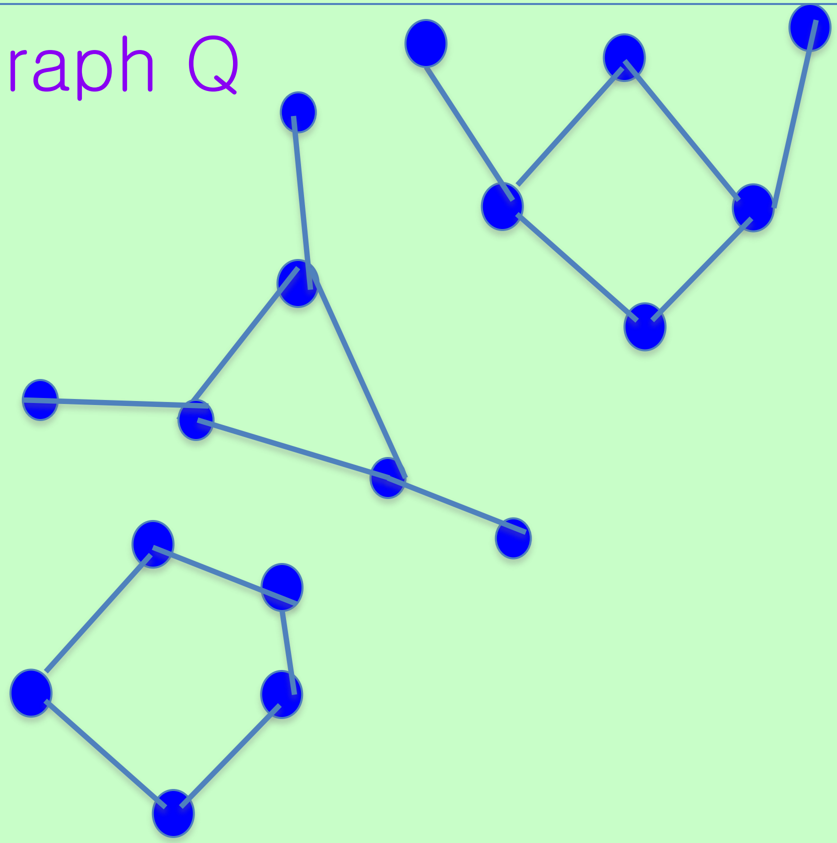
Overview

- Our Result
- Introduction
  - Complexity of computation
  - “Structure of graphs”
  - Algorithms: Dynamic Programming
  - Width parameters of graphs
  - Embeddings
- Obstructions
  - Embeddings: subgraphs and minors
  - Robertson–Seymour Theorem
  - Rankwidth, linear rankwidth, thread graphs

Obstructions to Linear Rankwidth 1

- (half) cube graph Q
- net graph N
- cycle C_5



Complexity of Computation

- Everybody agrees that NP–completeness most probably implies exponential complexity (some say that NP stands for "not polynomial", a subtle joke)
- When O(n) becomes **"sad face"… we get…**
- A recent hierarchy of complexity classes (W–hierarchy) which includes "fixed parameter tractable" (FPT) problems on the lowest level

Fixed Parameter Tractability

In general, the time complexity of an algorithm acting on input with length n and a parameter (say, bandwidth) k is $f(n,k)$

For a fixed k, this may be polynomial (in n) even though k may be in the exponent, $n^{g(k)}$

Of course, we would prefer k <u>not</u> in the exponent, as in $f(n,k)=h(k)n^c$

Structure of Graphs

- “Structure” of graphs:
  - Graph grammars
  - Hierarchical graphs
  - 2-structures
  - Modular decomposition
- Parsing structure of graphs
  - Series-parallel graphs
  - “Complement-reducible” graphs aka. Cographs
  - PQ-graphs

(Optimization) DP Algorithms

- Recursive structure of solutions:
  - Optimal solution is a function of optimal solutions to smaller (sub-) problems
- Dynamic Programming
  - A bottom-up traversal of the tree of sub-problems
  - or a parsing structure of the graph…

Width Parameters of Graphs

- Tree Decompositions
  - Treewidth: partial k-trees
  - Pathwidth: partial k-paths
  - Branchwidth,
  - Cliquewidth
  - Rankwidth
  - Linear rankwidth

Embeddings: Guest into Host

- Mapping of elements of G into elements of H
- Embeddings of a graph G in H
  - Topological: edges of G into internal vertex-disjoint paths of H
  - Minor: vertices of G into connected subsets of vertices of H
  - Vertex minor: vertices of G into vertices of H, modulo local equivalence

Description of Graph Classes via Obstructions

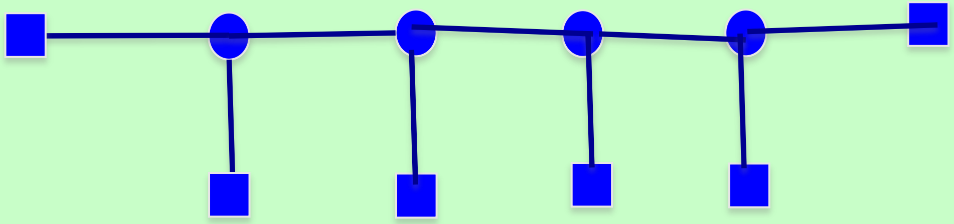
- Classes closed under embedding operation
- Minimal graphs outside the class of interest
- Examples of obstructions
  - Outerplanar graphs $\{K_4, K_{2,3}\}$
  - Planar graphs: $\{K_5, K_{3,3}\}$
  - Treewidth 3 graphs: $\{K_5, dW_4, M_8, M_{10}\}$
  - Linear rankwidth 1: $\{C_5, N, Q\}$

(Cubic) Tree Decomposition TD

- A cubic tree (internal nodes of degree 3) with leaf nodes labeled by elements of the graph
- Each tree branch partitions the graph elements into two blocks defined by the sets of disconnected leaves; evaluate the width function on this partition
- Maximum valuation (over all branches) determines the width of the decomposition
- The width of the graph is a minimum width over all decompositions.

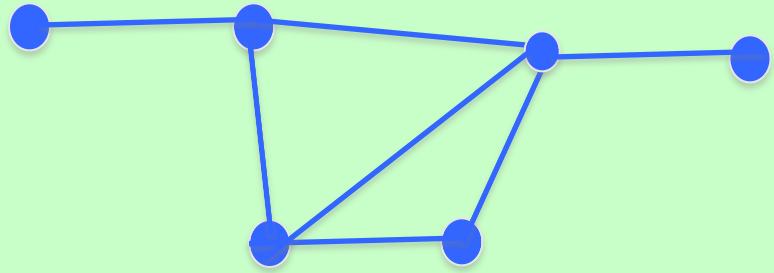
(linear width)

- tree of TD has linear structure: a caterpillar

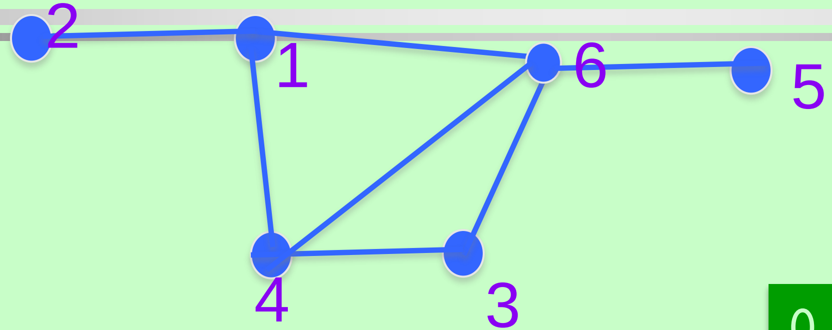


Rankwidth

- Leaves of the TD tree labeled by vertices of the graph
- Width of a branch is the rank of the adjacency matrix of the partition



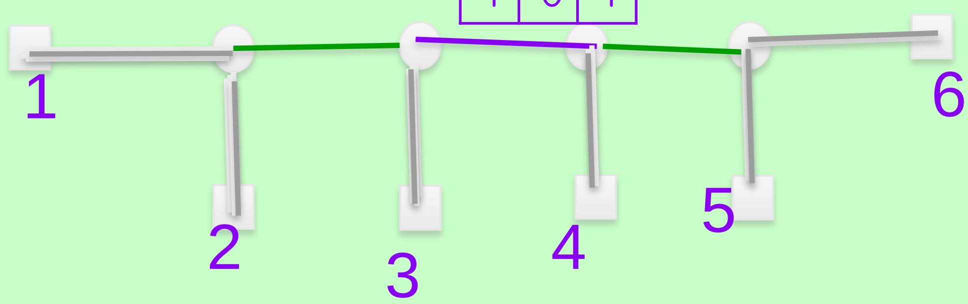
(example of width)



0	1	0	1
0	0	0	0

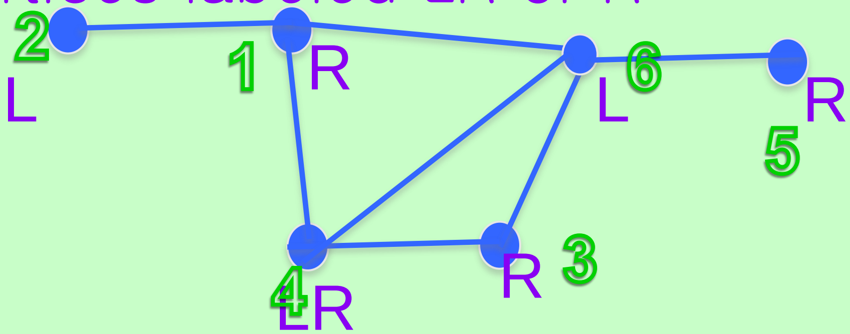
1	0	1
0	0	0
1	0	1

0	1
0	0
0	1
0	1



Thread Graphs

- [R. Ganian, IWOCA'10] A graph has linear rankwidth at most 1 iff it is a thread graph (thread blocks whose thread edges form a path)
- A graph is a thread block iff it has a *thread edge* (v_1,v_n) and an ordering of vertices $v_1,\cdots,v_n$ that admits *thread labeling* by L, R, LR, so that v_1 is labeled R, v_n is labeled L, and exactly vertices labeled L or LR are connected to *preceding* them vertices labeled LR or R

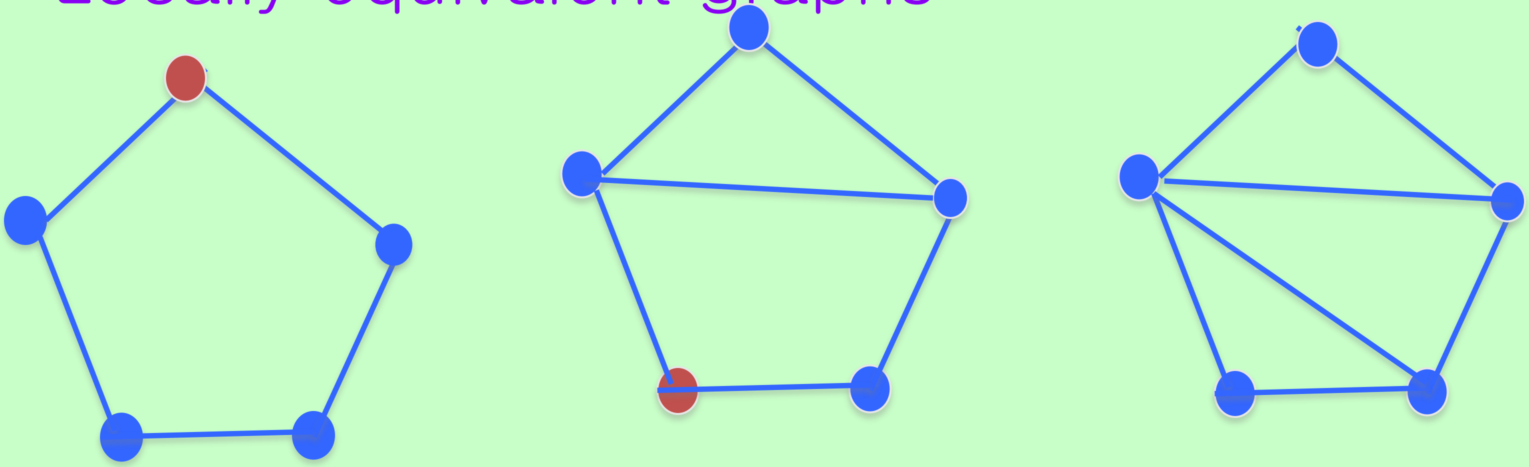


Vertex Minors

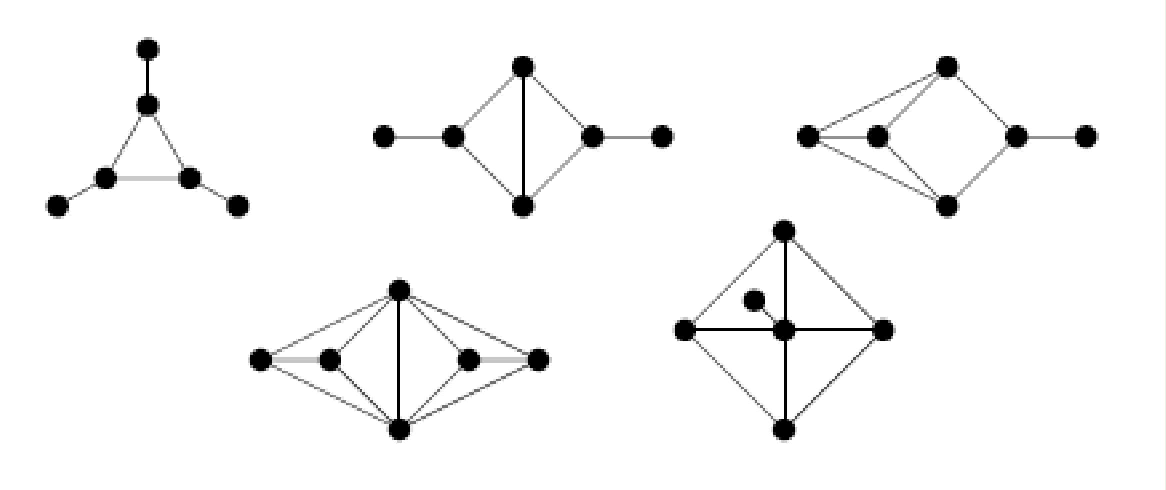
- Vertex minor: map vertices of G into vertices of H, modulo local equivalence
- Local equivalence, $G \sim G_* v$, where $_* v$ denotes
- L*ocal complementation* at vertex v of G, complementing adjacencies of the neighborhood of v in G.

Local complementation

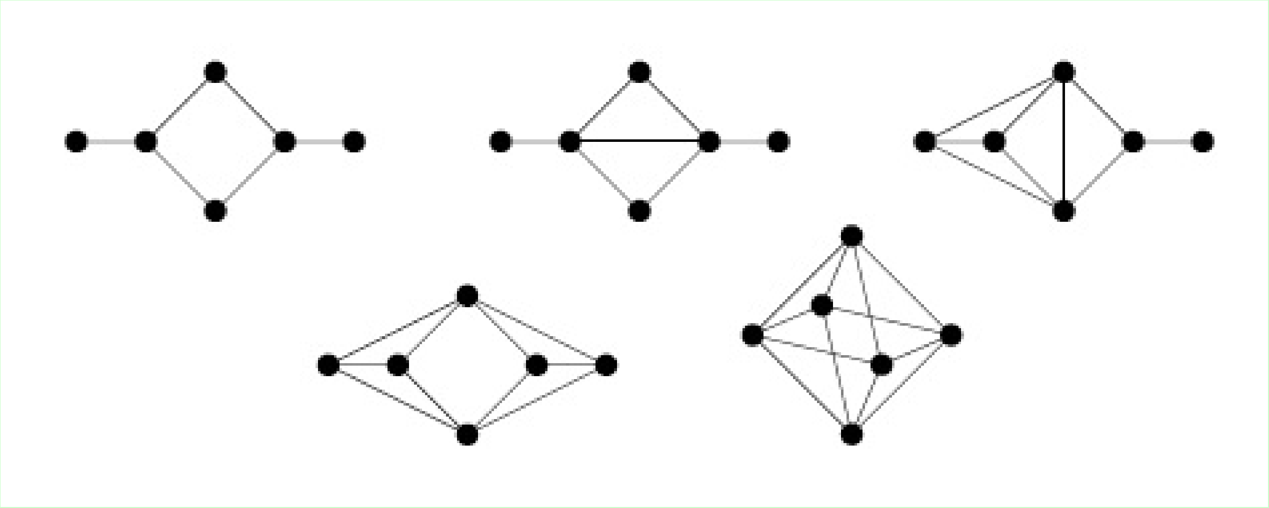
- Locally equivalent graphs



Graphs locally equivalent to N



Graphs locally equivalent to Q



Proof Sketch of Result

- graphs C_5, N, and Q are obstructions
  - neither is a thread graph
  - all their vertex minors are thread graphs
- every obstruction is connected
- in an obstruction, no cut-vertex separates more than two components
- case analysis for 0..3 pendant vertices

C_5, N, Q are the only obstructions