PRELIMINARY LECTURES ON KLEINIAN GROUPS

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In this two lectures, I shall present some backgrounds on Kleinian group theory, which, I hope, would be useful for understanding more advanced theory, which will be given by the main speakers in the latter half of the workshop. In the first part, I shall talk about the general theory, including definitions of basic notions in this field, and classical results. The basic references for this part are [4], [3] and [5]. In the second part, I shall focus on Thurston's theory of uniformisation (or hyperbolisation) of Haken manifolds, which is a beautiful but highly complicated theory. The best reference for this part is [2].

1. General theory

1.1. **Basic notions.** Kleinian groups are discrete subgroups of the Lie group $\mathrm{PSL}_2\mathbb{C}$, which is the group of linear fractional transformations of the Riemann sphere. This coincides with the group of orientation-preserving isometries of the hyperbolic space $\mathbb{H}^3$, where the Riemann sphere can be regarded as the sphere at infinity S^2_∞ in the Poincaré ball model of the hyperbolic space.

The elements of $\mathrm{PSL}_2\mathbb{C}$ are classified into three types: elliptic, parabolic, and loxodromic elements.

Definition 1.1. An element of $\mathrm{PSL}_2\mathbb{C}$ is said to be

(1) elliptic if it is conjugate to a matrix of the form $\begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$ with $|\omega| = 1$,

(2) parabolic if it is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and

(3) loxodromic if it is conjugate to a matrix of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ with $|\lambda| > 1$.

Since a Kleinian group is discrete, any elliptic element in a Kleinian group must be a torsion. Since a Kleinian group G acts on $\mathbb{H}^3$ by isometries, the action is properly discontinuous, and the quotient $\mathbb{H}^3/G$ is a hyperbolic 3-orbifold. Selberg's lemma says that any Kleinian

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group has a finite-indexed subgroup which is torsion-free. Therefore, any hyperbolic 3-orbifold has a finitely sheeted ramified covering which is a hyperbolic 3-manifold. This shows that usually it is sufficient to consider only torsion-free Keinian groups. In this talk, I always assume Kleinian groups to be torsion free.

A parabolic element acts on $\mathbb{H}^3$ without fixed points, but has a unique fixed point on S^2_∞. A loxodromic element has an axis in $\mathbb{H}^3$ on which the translation distance takes minimum, and its endpoints at S^2_∞ are the fixed points of the element on S^2_∞.

The following Margulis's lemma is used quite often in the theory of Kleinian groups.

Lemma 1.1 (Margulis's lemma). *There exists a universal constant ϵ_0 with the following properties. For any hyperbolic 3-manifold M, the set of points in M where the injectivity radii are less than ϵ_0 is a disjoint union of the following three types of sets.*

1. *Margulis tube: A tubular neighbourhood of a closed geodesic whose length is less than ϵ_0.*
2. *$\mathbb{Z}$-cusp-neighbourhood: The quotient of a horoball by a parabolic group isomorphic to $\mathbb{Z}$ fixing the tangent point.*
3. *$\mathbb{Z}\times\mathbb{Z}$-cusp-neighbourhood: The quotient of a horoball by a parabolic group isomorphic to $\mathbb{Z}\times\mathbb{Z}$ fixing the tangent point.*

This lemma also shows that any non-trivial abelian subgroup in a torsion-free Kleinian group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}\times\mathbb{Z}$, and that in the latter case the group consists of parabolic elements fixing the same point on S^2_∞.

Now, for the moment, I regard a Kleinian group as acting on the Riemann sphere (S^2_∞) by linear fractional transformations. I turn to consider the dynamics of the action of a Kleinian group on the Riemann sphere.

Definition 1.2. Let G be a Kleinian group. The limit set Λ_G of G is the closure of the set of fixed points on S^2_∞ of (non-trivial) elements of G. The complement of Λ_G is called the region of discontinuity of G.

As its name suggests, G acts on Λ_G properly discontinuously. Actually, Ω_G is the largest subset of S^2_∞ on which G acts properly discontinuously. Since G acts on Ω_G as conformal automorphisms, its quotient Ω_G/G is a Riemann surface. The following is a classical theorem by Ahlfors.

Theorem 1.2 (Ahlfors's finiteness theorem). *Let G be a finitely generated Kleinian group. Then the Riemann surface Ω_G/G is of finite* *type: that is, it has finitely many components each of which has finite genus and a finite number of punctures, without open ends.*

Ahlfors moreover conjectured that for every finitely generated Kleinian group, Λ_G either is the entire S^2_∞ or has null Lebesgue measure. This conjecture, called the Ahfors conjecture is known to be true now. Bonahon proved this conjecture for freely indecomposable Kleinian groups. (A group is said to be freely indecomposable when it cannot be decomposed into a non-trivial free product.) Canary then showed that this conjecture follows from Marden's tameness conjecture, which I shall explain later. Finally, Marden's tameness conjecture was solved by Agol and Calegari-Gabai independently. (Canary [1] is a very good exposition to learn these things.)

1.2. **Topological properties of hyperbolic 3-manifolds.** Now, I shall turn to a more topological aspect of the theory of Kleinian groups. From now on, I shall consider only *finitely generated Kleinian groups.*

Definition 1.3. Let $M = \mathbb{H}^3/G$ be a hyperbolic 3-manifold corresponding to a Kleinian group G. The convex core of M is the smallest convex submanifold of M that is a deformation retract.

We can construct a convex core explicitly as follows. Consider the limit set Λ_G, and take the convex hull H_G of the set consisting of all geodesics in $\mathbb{H}^3$ whose endpoints lie in Λ_G. This set H_G is G-invariant, and its quotient H_G/G is exactly the convex core of M.

A Kleinian group G and its corresponding hyperbolic 3-manifold $\mathbb{H}^3/G$ are said to be *geometrically finite* when the convex core of $\mathbb{H}^3/G$ has finite volume.

It is known that there are geometrically infinite Kleinian groups: for instance boundary groups constructed by Bers. Still, we can capture the topological property of the quotient manifold using the theory of 3-dimensional topology.

Theorem 1.3 (Scott, McCullough-Miler-Swarup). *Let M be an open 3-manfiold whose fundamental group is finitely generated. Then there is a compact 3-submanifold C of M such that the inclusion from C to M is a homotopy equivalence. The homeomorphism type of such a manifold is unique.*

Such a compact submanifold is called a *compact core* of M. In general, even if the fundamental group is finitely generated, M itself may not be homeomorphic to the interior of a compact 3-manifold. Marden's conjecture says that a *hyperbolic* 3-manifold with finitely generated fundamental group is always homeomorphic to the interior of a compact 3-manifold, hence the interior of its compact core. As was mentioned in the previous subsection, this conjecture is known to be true today.

1.3. **Deformation theory of Kleinian groups.** One of the most interesting part of the theory of Kleinian groups resides in the deformation theory. The notion of the deformations of Kleinian groups go back to the work of Ahlfors and Bers in 1960's.

Recall that a map f from a domain X in $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ is said to be quasi-conformal if $\mu(f) = \dfrac{\partial_{\bar{z}} f}{\partial_z f}$ has essential norm less than 1. For a quasi-conformal map f, we call $\dfrac{1+\|\mu(f)\|_\infty}{1-\|\mu(f)\|_\infty}$ the dilatation of f. The Teichmüller theory says that for any homeomorphic Riemann surfaces S and T with a homeomorphism $f : S \to T$, there is a unique quasi-conformal homeomorphism $g : S \to T$ homotopic to f with minimal dilatation. This map is called the Teichmüller map. For a Riemann surface S, its Teichmüller space is the set consisting of equivalence classes pairs (Σ, f), where f is a Teichmüller map from S to Σ and two pairs (Σ_1, f_1) and (Σ_2, f_2) are identified when $f_1 \circ f_2^{-1}$ is conformal. We can endow a topology where two points $(\Sigma_1, f_1), (\Sigma_2, f_2)$ are near when $f_1 \circ f_2^{-1}$ is near to a conformal homeomorphism.

A Kleinian group Γ is said to be a quasi-conformal deformation of G when there is a quasi-conformal homeomorphism $f : S^2_\infty \to S^2_\infty$ such that $\Gamma = fGf^{-1}$. The set of all quasi-conformal deformations of G is denoted by $QC(G)$, where two deformations are identified when they are conjugate, and the topology is the one induced from the representation space of G into $\mathrm{PSL}_2\mathbb{C}$ modulo conjugacy.

The following result due to Ahlfors, Bers, Maskit, Kra, Marden is essential.

Theorem 1.4. *There is a ramified covering map $qf : \mathcal{T}(\Omega_G/G) \to QC(G)$.*

Thus, unless $\mathcal{T}(\Omega_G/G)$ is trivial, there are non-trivial deformations of G. This should be contrasted with the extreme case when $\mathbb{H}^3/G$ has finite volume, which implies that Ω_G is empty in particular. Then Mostow's rigidity theorem says the following.

Theorem 1.5 (Mostow)**.** *Let G be a Kleinian group such that $\mathbb{H}^3/G$ has finite volume. Then any Kleinian group Γ that has an isomorphism $\phi : G \to \Gamma$ preserving the parabolicity in both directions is realised as a conjugation in* $\mathrm{PSL}_2\mathbb{C}$.

In the case when G is a Fuchsian group, its quasi-conformal deformations are called quasi-Fuchsian groups. The map qf is a homeomorphism from $\mathcal{T}(S) \times \mathcal{T}(S)$ in this case, where S is a Riemann surface corresponding to G.

We can consider a larger space of deformations:

Definition 1.4. For a Kleinian group G, the deformation space of G, denoted by $AH(G)$ is defined to be the set of faithful discrete representations of G into $\mathrm{PSL}_2\mathbb{C}$ preserving the parabolicity, modulo conjugacy.

Mostow's rigidity theorem says this space $AH(G)$ is trivial when $\mathbb{H}^3/G$ has finite volume. It was proved by Sullivan that when G is geometrically finite, $QC(G)$ is an open subset of $AH(G)$. The Bers-Sullivan-Thurston density conjecture says that for a geometrically finite Kleinian group G, the quasi-conformal defomration space is dense in the entire the deformation space. This conjecture is known to be true today by work of Thurston, Kleineidam-Souto, Bromberg, Brock-Bromberg, Lecuire, Kim-Lecuire-Ohshika, Namazi-Souto and Ohshika.

2. Thurston's uniformisation theorem for Haken manifolds

In this second lecture, I shall focus on Thurston's uniformisation theorem for Hekan manifolds. I start with some basic notions in 3-manifold topology.

2.1. **Basic notions for 3-manifolds.** In this talk, a 3-manifold is always assumed to be C^∞ and orientable. We say that a 3-manifold M is irreducible when every 2-sphere embedded in M bounds a ball. By the prime-decomposition theorem due to Kneser and Milnor, there is a unique prime decomposition of M into irreducible manifolds and $S^2 \times S^1$. (A manifold is called prime when it does not have a separating 2-sphere not bounding a ball.) From now on, I shall only deal with irreducible 3-manifolds.

An orientable embedded surface S in M is said to be incompressible when the inclusion induces a monomorphism between the fundamental groups. I always assume surfaces to be orientable from now on when I talk about embedded surfaces in 3-manifolds. A 3-manifold is said to be *Haken* when it is irreducible and has an incompressible surface.

Haken manifolds had been called irreducible sufficiently large 3-manifolds up to 1970's. As was shown in fundamental work by Waldhausen, a Haken manifold admits a hierarchy by which the manifold is reduced to a union of balls by being cut along incompressible surfaces. The existence of hierarchies shows that an inductive argument works for Haken manifolds, which was the main technique in Waldhausen's work showing homotopy equivalent closed Haken manifolds are homeomorphic.

Let M be a Haken manifold. The theory of Jaco-Shalen-Johannson says that there is a unique family of disjoint incompressible tori $\{T_j\}$ and annuli $\{A_k\}$ in M such that each component of $M \setminus (\cup_j T_j \cup \cup A_k)$ is the interior of either a Seifert fibred manifold or an atoroidal manifold. A 3-manifold M is said to be atoroidal when every incompressible torus in M is homotopic to a boundary component.

2.2. **Uniformisation theorem.** The following theorem is Thurston's unifomisation theorem.

Theorem 2.1. *Let M be a compact irreducible atoroidal Haken manifold. Then the interior of M admits a geometrically finite hyperbolic metric.*

This means that any Haken manifold is decomposed into Seifert fibred manifolds and hyperbolic manifolds by cutting along disjoint incompressible tori and annuli. The proof of this theorem is very long and complicated. Actually Thurston planned to write a paper consisting of seven parts to show this. In this talk, I shall present one part of his argument, which I think is the most essential.

2.3. **Pasting along quasi-Fuchsian groups.** The proof of Thurston's uniformisation theorem uses the following theorem, which is a special case of a more general theorem called Maskit's combination theorem in each step of induction based on a hierarchy.

Theorem 2.2 (Maskit)**.** *Let M_1 and M_2 be two Haken manifolds whose interiors admit geometrically finite hyperbolic metrics. Suppose that there are incompressible boundary components S_1 of M_1 and S_2 of M_2 with a homeomorphism $h : S_1 \to S_2$. Let M be a Haken manifold obtained by glueing M_1 and M_2 identifying S_1 and S_2 via h. Let G_{S_1} and G_{S_2} be quasi-Fuchsian groups corresponding to subgroups $\pi_1(S_1) \subset \pi_1(M_1)$ and $\pi_1(S_2) \subset \pi_1(M_2)$. Suppose further that G_{S_1} and G_{S_2} are conjugate in $\mathrm{PSL}_2\mathbb{C}$. Then there is a geometrically finite hyperbolic metric on $\mathrm{Int}M$ such that the coverings of M associated to $\pi_1(M_1)$ and $\pi_1(M_2)$ give original hyperbolic metrics on $\mathrm{Int}M_1$ and $\mathrm{Int}M_2$ respectively.*

To illustrate Thurston's argument, now I focus on the last step of the induction for the case when M is a closed manifold. Actually, there are two different situations for this special case. The one is when M is a surface bundle over S^1, and the other is when M is not "fibred".

I shall only consider the latter case, assuming further M_1 and M_2 are acylindrical: i.e. we assume that they contain no essential annuli. Since this is the last step and we assumed that M is closed, we have $\partial M_1 = S_1$ and $\partial M_2 = S_2$.

Recall that we can consider the spaces of quasi-conformal deformations of the original geometrically finite hyperbolic metrics of M_1 and M_2, which we denote by $QC(M_1)$ and $QC(M_2)$ respectively. They are open subsets of the entire deformation spaces $AH(M_1)$ and $AH(M_1)$ respectively.

Let x be a point in $QC(M_1)$. Then we consider the quasi-Fuchsian group $G_{S_1}(x)$ corresponding to $\pi_1(S_1)$ in $(\mathrm{Int}M_1, x)$. Now, recall that the space of quasi-Fuchsian groups are parametrised by $\mathcal{T}(S_1) \times \mathcal{T}(S_1)$. One of the parameter corresponds to the conformal structure appearing as the conformal structure at infinity of M_1, whereas the other is hidden inside M_1. We pick up the second coordinate, the hidden one, and denote it by $q(x)$. Regarding $q(x)$ as the conformal structure at infinity for M_2, identifying S_1 with S_2 via h, we get a point $r(x) \in QC(M_2)$. We consider the quasi-Fuchsian group $G_{S_2}(q(x))$ corresponding to $\pi_1(S_1)$ in $(\mathrm{Int}M_2, r(x))$, and take the hidden conformal structure, which we denote by $s(x)$. What we need to apply Maskit's theorem to glue the hyperbolic structures is the condition $x = s(x)$. Therefore, what we have to do is look for a fixed point for the map $s : \mathcal{T}(S_1) \to T(S_1)$.

McMullen proved that this map s, which is called the skinning map, has a fixed point by showing that s is distance-decreasing. Thurston's original proof is more complicated. He showed that s has a fixed point if we consider the larger space $AH(M_1)$ instead of $QC(M_1)$ since $AH(M_1)$ is compact in the present case, and then showed that the fixed point cannot lie outside $QC(M_1)$ using the "covering theorem" which says that a covering of a geometrically infinite end is always finite-sheeted.

References

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