Total fractional colorings of graphs with large girth

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Overview

- Graph colorings—basic notions
- Fractional graph parameters
- Problem and our results
- Main proof idea
**Vertex colorings**

- two adjacent vertices must receive distinct colors
- chromatic number $\chi(G)$
- $\chi(G) \leq \Delta + 1$

Brooks’ theorem (1941): $\chi(G) \leq \Delta$ for a connected graph $G$ unless $G$ is complete or an odd cycle
Edge colorings

- two incident edges must receive distinct colors
- chromatic index $\chi'(G)$
- Vizing’s theorem (1964): $\chi'(G) \in \{\Delta, \Delta + 1\}$
  Holyer (1981): NP-complete to decide between the two values
Total colorings

- coloring of vertices and edges
  any two adjacent/incident elements must receive distinct colors
- total chromatic number $\chi_t(G)$
- Behzad’s conjecture (1965): $\chi_t(G) \leq \Delta + 2$
  Molloy and Reed’s bound (1998): $\chi_t(G) \leq \Delta + 10^{28}$
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Fractional colorings

- coloring = partitioning into disjoint color classes
- fractional coloring
  each color class has weight, they need not be disjoint
  every element in classes of total weight at least one
  minimizing the total weight of all color classes
- total fractional chromatic number $\chi_f(G)$
- weights only zero or one $\Rightarrow$ coloring
Example

\[ \chi_{t,f} = 3 \text{ and } \chi_t = 4 \]
Basic results

- clearly, $\chi_f(G) \leq \chi(G)$, $\chi'_f(G) \leq \chi'(G)$ and $\chi_{t,f}(G) \leq \chi_t(G)$
- the gap between $\chi_f(G)$ and $\chi(G)$ can be arbitrary
  $\chi(G)$ can be arbitrary and $\chi_f(G) \leq 2 + \varepsilon$
- $\chi_f(G) \leq 2.416$ if $G$ is cubic and has large girth
  examples of cubic graphs with large girth with $\chi_f(G) \geq 2.196$
- $\chi'_f(G) = 3$ for cubic bridgeless graphs
- $\Delta \leq \chi'_f(G) \leq \Delta + \varepsilon$ for graphs with large girth
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PREVIOUS RESULTS

• Behzad’s conjecture is fractionally true
  Kilakos and Reed (1993): $\chi_{t,f}(G) \leq \Delta + 2$

• When does the equality hold?
  Ito, Kennedy and Reed (2009): only if $G$ is $K_{2n}$ or $K_{n,n}$

• What about large girth?
Reed’s conjecture

Let $\Delta$ be an integer. For every $\varepsilon > 0$, there exists $g$ such that every graph $G$ with maximum degree $\Delta$ and girth at least $g$ has total fractional chromatic number at most $\Delta + 1 + \varepsilon$. 
Our results

• Theorem (Kardoš, Král’, Sereni):
  Let $\Delta$ be an integer. For every $\varepsilon > 0$, there exists $g$ such that every graph $G$ with maximum degree $\Delta$ and girth at least $g$ has total fractional chromatic number at most $\Delta + 1 + \varepsilon$.

• Theorem (Kaiser, King, Král’):
  Let $\Delta \in \{3, 4, 6, 8, \ldots\}$. There exists $g$ such that every graph $G$ with maximum degree $\Delta$ and girth at least $g$ has total fractional chromatic number equal to $\Delta + 1$. 
OVERVIEW

• Graph colorings—basic notions
• Fractional graph parameters
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Decomposition into paths

- \( \chi_{t,f}(G) \leq \alpha \iff \exists \) probability distribution on independent sets, each element included with probability at least \( 1/\alpha \)

- exposition restricted to cubic bridgeless case for simplicity (3 slides)
  \( \alpha = 4 + \varepsilon \)

- there exists a distribution on perfect matchings such that each edge included with probability \( 1/3 \)

- remove vertices at distance \( g/100 \) from the complementary 2-factor collection of paths of length at most \( g/100 \)
  each edge included with probability at least \( 2/3 - 200/g \)
  each vertex with probability \( 1 - 100/g \)
Colorings by levels

- split paths into 10 levels
- sweep paths in the 1st level, then in the 2nd level, etc.
  include vertices and edges greedily into a total independent set
- if a vertex can be included, include it with probability $1 - \xi$
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SUMMARY FOR CUBIC GRAPHS

• levels guarantee mutual independence of neighbors for each path
• for a suitable choice of $\xi > 0$
  each vertex on a path included with probability $1/4$
  each edge in a path included with probability $3/8$
• considering the distribution over perfect matchings:
  each vertex included with probability $1/4 - 25/g$
  each edge included with probability $1/4 - 75/g$
• a better resolution of coloring conflicts at the ends of the paths
  needed to obtain $\chi_{t,f} = 4$ for large $g$
Differences for general graphs

- if $\Delta = 4, 6, 8, \ldots$, uniform coverings by 2-factors exist
- if $\Delta = 5, 7, 9, \ldots$, uniform coverings by 2-factors exist assuming the graph is cyclically $(\Delta - 1)$-edge-connected
- a stronger result on extending precolorings can be proven
- if the graph is not cyclically $(\Delta - 1)$-edge-connected, split the graph into two pieces, one inclusion-wise minimal the minimal piece is cyclically $(\Delta - 1)$-edge-connected induction on the bigger piece and extend to the smaller one
Thank you for your attention!