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"Sheaves on toric Calabi-Yau hypersurfaces"

$Y \subset \mathbb{P}^n$ hypersurface $Y = Z(W)$, $W \in \mathbb{C}[x_0 \ldots x_n]$

$S_W = \mathbb{C}[x_0 \ldots x_n]/W$ homog. coord. ring of Y

= functions on the affine cone $CY \subset \mathbb{C}^{n+1}$

$$Y \;=\!=\; (\underline{CY} \setminus \underline{\underline{0}})/\underline{\underline{\mathbb{C}^*}}$$

sheaves on Y $=\!=$ $\underline{\text{graded}}$ $\underline{\text{modules over } S_W}$ $\underline{\text{up to finite dimensional modules}}$

$D^b(Y) =$ finite complexes of f.g. graded S_W-modules / complexes of finite dimensional modules and quasi-isomorphism

$D^b_{sing}(S_W) =$ finite complexes of f.g. graded S_W-modules / complexes of projective modules and quasi-isomorphism

Similar definition Y a variety

$D^b_{sing}(Y) =$ finite complexes of coherent sheaves on Y / complexes of vector bundles. and quasi-isomorphism

$\boxed{\text{If } W: X \to \mathbb{C} \text{ write } D^b_{sing}(W: X \to \mathbb{C}) \text{ for } D^b_{sing}(W^{-1}(0))}$

Why D^b_{sing}?

If Y is smooth, for any coherent sheaf $\mathcal{F}$ $\exists$ a $\underline{\text{finite}}$ resolution by vector bundles

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to \mathcal{F} \to 0$$

so in $D^b_{sing}(Y)$,

$$\mathcal{F} \underset{qis}{\sim} P_n \to \ldots P_1 \to P_0 \sim 0$$

projective mod or vector bundles

Studied by Orlov, Krause

E.g. when X has isolated singularities $P_1 \dots P_k$

$$D^b_{sing}(X) = \bigoplus_i \left(\begin{array}{c} \text{local version of } D^b_{sing} \\ \text{at } P_i \end{array} \right)$$

<u>Expected statement</u> $X = Z(W) \subset \mathbb{C}^{n+1}$ has origin as the only singular point

then

$$HH_\bullet\left(D^b_{sing}(X)\right) \simeq \text{Jacobian ring of origin} := \mathbb{C}[x_0 \dots x_n] \Big/ W, \partial W / \partial x_i$$

$\uparrow$ Hochschild Homology (see B. Keller)

<u>Theorem (Orlov)</u>

If $\deg W = n+1$ then

$$D^b(Y) \simeq D^b_{sing}(S_W)$$

(Actually this is a particular case of a more general statement proved by Orlov. Observe that $\deg W = n+1 \iff Y$ is Calabi-Yau).

<u>Want</u> Study $D^b(Y)$ when Y is Calabi-Yau complete intersection in a general projective smooth toric variety $\mathbb{P}$ (also when $Y, \mathbb{P}$ are quasi-smooth)

Original algebraic proof of Orlov uses a functor

$\mathcal{F}$ bundle $\longrightarrow$ complex of graded S_W-modules with i-th cohomology $\bigoplus_{n \geq 0} H^i(Y, \mathcal{F}(n))$

For general toric $\mathbb{P}$ need to consider $\bigoplus_{\mathcal{L} \in \operatorname{Pic}(\mathbb{P})} H^i(Y, \mathcal{F} \otimes \mathcal{L})$, and it is hard to replace the condition "$\mathcal{L} \geq 0$"

<u>Geometric Restatement</u> based on

<u>Theorem (Umut)</u> $\mathbb{P}$ smooth projective $W \in H^0(\mathbb{P}, K_{\mathbb{P}}^{\vee})$, $Y = Z(W)$. Consider the function $\widetilde{W} : \Omega_{\mathbb{P}}^n \to \mathbb{C}$, where $\Omega_{\mathbb{P}}^n$ is the total space of the canonical bundle ($n = \dim_{\mathbb{C}} \mathbb{P}$) and $\widetilde{W}$ is obtained by pairing with W fiberwise.

Then $\quad \mathcal{D}^b(Y) = \mathcal{D}^b_{sing}(\widetilde{W} : \Omega_{\mathbb{P}}^n \to \mathbb{C})$

(Again, this is a particular case of a more general statement).

<u>Idea behind it</u> generalize classical BGG correspondence

$$D^b(\Lambda^\bullet(V)) \simeq D^b(\operatorname{Sym}^\bullet(V^\vee))$$

$$\Lambda^1(V) \leftrightarrow K_{\mathbb{P}} \qquad \operatorname{Sym}^\bullet(V^\vee) \leftrightarrow \operatorname{Sym}^\bullet(K^\vee) = \text{functions on } \Omega_{\mathbb{P}}^n$$

$$\Lambda^0(V) \leftrightarrow \mathcal{O}_{\mathbb{P}}$$

This convert's Orlov's Theorem into

$$D^b_{sing}(\widetilde{W} : \Omega^n_{\mathbb{P}^n} \to \mathbb{C}) \xrightarrow{\sim} D^b_{sing}(W : \mathbb{C}^{n+1} \to \mathbb{C})$$

Some grading issues.

the functions are essentially the same but the spaces are different

Spaces are different

$$\begin{tikzcd} \Omega^n_{\mathbb{P}^n} \arrow[r] & \mathbb{C}^{n+1}/\mathbb{Z}_{n+1} & {[\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}]} \arrow[l] \end{tikzcd}$$

orbifold or DM stack.

contracts zero section to the singular point, isomorphism away from it.

<u>Note on orbifolds:</u>

bundles on $[\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}]$ = $\mathbb{Z}_{n+1}$-equivariant bundles on $\mathbb{C}^{n+1}$

bundles on $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}$ = $\mathbb{Z}_{n+1}$-equivariant bundles for which $\mathbb{Z}_{n+1}$ acts trivially on the fiber over $0 \in \mathbb{C}^{n+1}$

<u>McKay Correspondence</u>:

$$D^b(\Omega^n_{\mathbb{P}^n}) \simeq D^b([\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}])$$

two crepant resolutions of the same space

(similar to Bridgeland-King-Reid)

<u>Theorem</u> (Pecharich) Suppose

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] & \\ & Z \arrow[r, "W"] & \mathbb{C} & \end{tikzcd}$$

two resolutions (may be orbifolds)

and $\mathcal{F}^\cdot$ is a complex on $X \times Y$ (cohomology on $X \times_Z Y$) s.t.

$$D^b(X) \longrightarrow D^b(Y)$$
$$G \longrightarrow (\pi_Y)_*(\pi_X^*(G) \otimes \mathcal{F}^\cdot)$$

is an equivalence. Then for $W: Z \to \mathbb{C}$ we have

$$D^b_{sing}(X \to Z \xrightarrow{W} \mathbb{C}) \simeq D^b_{sing}(Y \to Z \xrightarrow{W} \mathbb{C})$$

This gives Orlov's Theorem when combined with the result of Umut.

Generalization to smooth toric case

$\mathbb{P}$ = projective smooth toric variety of $\dim_{\mathbb{C}} = n$

$\mathbb{P} = (\mathbb{C}^p \setminus B)/G$ where

Cox-Batyrev construction

p = # of 1-dim cones in the fan of $\mathbb{P}$

B = union of some coordinate subspaces of codim ≥ 2

$G = \mathrm{Hom}(\mathrm{Pic}(\mathbb{P}), \mathbb{C}^*) = (\mathbb{C}^*)^{p-n}$ if $\mathbb{P}$ smooth projective

G acts on $\mathbb{C}^p$ with some weights $\alpha_1, \dots, \alpha_p$.

to get $\Omega^n_{\mathbb{P}}$ consider $(\mathbb{C}^p \setminus B) \times \mathbb{C}/G$ additional weight is $-(\alpha_1 + \dots + \alpha_p)$

Assume <u>$\mathbb{P}$ is Fano</u> then

$\Omega^n_{\mathbb{P}} \to Z = \mathrm{Spec}\,(H^0(\Omega^n_{\mathbb{P}}, \mathcal{O}))$ is a proper birational map

Need another resolution of Z.

Define G_K as $\mathrm{Ker}\,(G \xrightarrow{} \mathbb{C}^*)$

$\uparrow$ of weight $-(\alpha_1 + \dots + \alpha_p)$

Consider "$\mathbb{C}^p/G_K$" = "$\mathbb{C}^p \times \mathbb{C}^*/G$" and take $Y = Z(W)$ for $W \in H^0(\mathbb{P}, K^\vee_{\mathbb{P}})$. We have

<u>Theorem</u> (–, J. Pecharich)

$$D^b(Y) \simeq D^b_{\mathrm{sing}}\left(\text{"}\mathbb{C}^p/G_K\text{"} \to Z \xrightarrow{W} \mathbb{C}\right)$$

"…" because one has to make additional choice (throw out some points) to define the quotient $/G$.

Example 0

$\mathbb{P} = \mathbb{P}^n$, $G_K = \mathbb{Z}_{n+1}$ $[\mathbb{C}^n/\mathbb{Z}_{n+1}]$ as orbifold, no additional choices.

In the next two examples $G = \mathbb{C}^* \times \mathbb{C}^*$

$\mathbb{P} = (\mathbb{C}^4 \setminus B)/_{\mathbb{C}^* \times \mathbb{C}^*}$, $\mathbb{C}^4$ has coord (x, y, u, v)

Example 1 $\mathbb{P} = \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

weights of G-action on $\mathbb{C}^4 \times \mathbb{C}$: $\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$

$$G_K = \{(\xi, \zeta) \mid \xi\zeta^2 = 1\} = \{(\xi, \zeta) \mid \xi = \zeta^{-2}\} = \mathbb{C}^*$$

$Z_1 = \{x = y = 0\}$ $\quad$ G_K acts on $\mathbb{C}^4$
$Z_2 = \{u = v = 0\}$ $\quad$ with weights $(-2, -2, 1, 3)$

$(\mathbb{C}^4 \setminus Z_1)/_{\mathbb{C}^*}$ and $(\mathbb{C}^4 \setminus Z_2)/_{\mathbb{C}^*}$

two quotients, differ by a flip.

Example 2 $\mathbb{P} = \mathbb{P}^1 \times \mathbb{P}^1$ weights $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right)$

$$G_K = \{(\xi, \zeta) \mid \xi^2\zeta^2 = 1\} = \mathbb{C}^* \times \mathbb{Z}_2$$
$$\xi = \pm\zeta^{-1} \qquad \zeta \quad \pm 1$$

G_K acts on $\mathbb{C}^4$ by $(\zeta^{-1}, \zeta^{-1}, \pm\zeta, \pm\zeta)$

$(\mathbb{C}^4 \setminus Z_1)/_{\mathbb{C}^* \times \mathbb{Z}_2}$ $\qquad$ $(\mathbb{C}^4 \setminus Z_2)/_{\mathbb{C}^* \times \mathbb{Z}_2}$

differ by a flop.

THE END.