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①

"Sheaves on toric Calabi-Yau hypersurfaces"

$Y \subset \mathbb{P}^n$ hypersurface $Y = Z(W)$, $W \in \mathbb{C}[x_0, \dots, x_n]$

$S_W = \mathbb{C}[x_0, \dots, x_n]/W$ homog. coord. ring of Y
= functions on the affine cone $CY \subset \mathbb{C}^{n+1}$

$$Y \cong \underline{(CY - 0)} / \underline{\underline{\mathbb{C}^*}}$$

sheaves on $Y \cong$ graded modules over S_W
up to finite dimensional modules

$D^b(Y) =$ finite complexes of f.g. graded S_W -modules / complexes of finite dimensional modules and quasi-isomorphism

$D_{\text{sing}}^b(S_W) =$ finite complexes of f.g. graded S_W -modules / complexes of projective modules and quasi-isomorphism

Similar definition Y a variety.

$D_{\text{sing}}^b(Y) =$ finite complexes of coherent sheaves on Y / complexes of vector bundles and quasi-isomorphism

Why D_{sing}^b ? IF $W: X \rightarrow \mathbb{C}$ write $D_{\text{sing}}^b(W: X \rightarrow \mathbb{C})$ for $D_{\text{sing}}^b(W^{-1}(0))$

IF Y is smooth, for any coherent sheaf \mathcal{F} \exists a finite resolution by vector bundles

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathcal{F} \rightarrow 0$$

so in $D_{\text{sing}}^b(Y)$,

$$\mathcal{F} \sim_{\text{qis}} P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \sim 0$$

projective mod
or vector bundles

Studied by Orlov, Krause

E.g. when X has isolated singularities $P_1 \dots P_k$

$$D_{\text{sing}}^b(X) = \bigoplus_i (\text{local version of } D_{\text{sing}}^b \text{ at } P_i)$$

Expected statement $X = Z(W) \subset \mathbb{C}^{n+1}$
has origin as the only singular point

then

$$HH_{\text{ho}}^0(D_{\text{sing}}^b(X)) \simeq \text{Jacobian ring of origin} := \mathbb{C}[x_0 \dots x_n] / \langle \frac{\partial W}{\partial x_i} \rangle$$

Hochschild Homology (see B. Keller)

Theorem (Orlov)

If $\deg W = n+1$ then

$$D^b(Y) \simeq D_{\text{sing}}^b(S_W)$$

(Actually this is a particular case of a more general statement proved by Orlov. Observe that $\deg W = n+1 \iff Y$ is Calabi-Yau).

Want Study $D^b(Y)$ when Y is Calabi-Yau complete intersection in a general projective smooth toric variety \mathbb{P} (also when Y, \mathbb{P} are quasi-smooth)

Original algebraic proof of Orlov uses a functor

$$\mathcal{F} \text{ bundle} \longrightarrow \text{complex of graded } S_W\text{-modules with } i\text{-th cohomology } \bigoplus_{n \geq 0} H^i(Y, \mathcal{F}(n))$$

For general toric P need to consider

$\bigoplus H^i(Y, F \otimes L)$, and it is hard to $\mathcal{L} \in \text{Pic}(P)$
replace the condition " $L \geq 0$ "

Geometric Restatement based on

Theorem (Umut) P -smooth projective
 $W \in H^0(P, K_P^V)$, $Y = Z(W)$. Consider the
function $\tilde{W}: \Omega_P^n \rightarrow \mathbb{C}$, where Ω_P^n is the total
space of the canonical bundle ($n = \dim_{\mathbb{C}} P$)
and \tilde{W} is obtained by pairing with W fiberwise.

Then $\mathcal{D}^b(Y) = \mathcal{D}_{\text{sing}}^b(\tilde{W}: \Omega_P^n \rightarrow \mathbb{C})$

(Again, this is a particular case of a more general statement).

Idea behind it
correspondence

generalize classical BGG
 $\mathcal{D}^b(\Lambda^*(V)) \simeq \mathcal{D}^b(\text{Sym}^*(V^V))$
 $\Lambda^1(V) \leftrightarrow K_P$ $\text{Sym}^*(V^V) \xrightarrow{\sim} \text{Sym}^*(K^V) =$ functions on Ω_P^n
 $\Lambda^0(V) \leftrightarrow \mathcal{O}_P$

This convert's Orlov's Theorem into

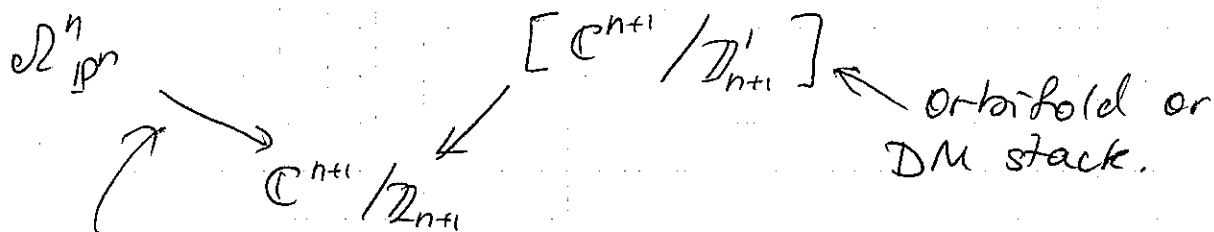
$$\mathcal{D}_{\text{sing}}^b(\tilde{W}: \Omega_{P^n}^1 \rightarrow \mathbb{C}) \simeq \mathcal{D}_{\text{sing}}^b(W: \mathbb{C}^{n+1} \rightarrow \mathbb{C})$$

Some grading issues.

the functions are essentially the same
but the spaces are different

Spaces are different

(4)



contracts zero section to the singular point, isomorphism away from it

Note on orbifolds:

bundles on $[\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}] = \mathbb{Z}_{n+1}$ -equivariant bundles on \mathbb{C}^{n+1}

bundles on $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1} = \mathbb{Z}_{n+1}$ -equivariant bundles for which \mathbb{Z}_{n+1} acts trivially on the fiber over $0 \in \mathbb{C}^{n+1}$

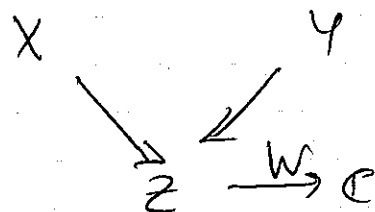
Mukai Correspondence

$$D^b(\mathcal{O}_{\mathbb{P}^n}^n) \simeq D^b([\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}])$$

two crepant resolutions of the same space

(similar to Bridgeland-King-Reid)

Theorem (Pechenich)
Suppose



two resolutions (may be orbifolds)

and \mathcal{F}^\bullet is a complex on $X \times Y$ (cohomology on

$X \times_{\mathbb{Z}} Y$) s.t.

$$D^b(X) \rightarrow D^b(Y)$$

$$G \rightarrow (\pi_Y)_* (\pi_X^*(\mathcal{F}) \otimes \mathcal{F}^\bullet)$$

is an equivalence. Then for $w: Z \rightarrow \mathbb{C}$ we have

$$D_{\text{sing}}^b(X \rightarrow Z \xrightarrow{w} \mathbb{C}) \simeq D_{\text{sing}}^b(Y \rightarrow Z \xrightarrow{w} \mathbb{C})$$

This gives Orlov's Theorem when combined with the result of Umut. (5)

Generalization to smooth toric case

P = projective smooth toric variety of $\dim_{\mathbb{C}} = n$

$P = (\mathbb{C}^p \setminus B) / G$ where
Cox-Batyrev construction

p = # of 1-dim cones in the fan of P
 B = union of some coordinate subspaces of $\text{codim} \geq 2$

$G = \text{Hom}(\text{Pic}(P), \mathbb{C}^{\times})$ if P is smooth projective
 $= (\mathbb{C}^{\times})^{p-n}$

G acts on \mathbb{C}^p with some weights $\alpha_1, \dots, \alpha_p$.

to get Ω_P^n consider $(\mathbb{C}^p \setminus B) \times \mathbb{C} / G$ additional weight is $-(\alpha_1 + \dots + \alpha_p)$

Assume P is Fano then

$\Omega_P^n \rightarrow Z = \text{Spec}(H^0(\Omega_P^n, \mathcal{O}))$ is a proper birational map

Need another resolution of Z .

Define G_K as $\text{Ker} \left(\begin{array}{c} G \rightarrow \mathbb{C}^{\times} \\ \uparrow \\ \text{of weight } -(\alpha_1 + \dots + \alpha_p) \end{array} \right)$

Consider " \mathbb{C}^p / G_K " = " $\mathbb{C}^p \times \mathbb{C}^{\times} / G$ " and take $Y = Z(W)$ for $W \in H^0(P, K_P^{\vee})$. We have

Theorem (-, J. Rehanich)

$$D^b(Y) \cong D^b_{\text{sing}}(\mathbb{C}^p / G_K \rightarrow Z \xrightarrow{W} \mathbb{C})$$

"..." because one has to make additional choice (throw out some points) to define the quotient $/G$. (6)

Example 0

$P = \mathbb{P}^n, G_k = \mathbb{Z}_{n+1} \quad [\mathbb{C}^n / \mathbb{Z}_{n+1}]$ as orbifold, no additional choices.

In the next two examples $G = \mathbb{C}^\times \times \mathbb{C}^\times$

$P = (\mathbb{C}^4 - B) / (\mathbb{C}^\times \times \mathbb{C}^\times), \mathbb{C}^4$ has coord (x, y, u, v)

Example 1 $P = \mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

weights of G -action on $\mathbb{C}^4 \times \mathbb{C}$: $\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array} \right) \begin{array}{l} -1 \\ 2 \end{array}$

$G_k = \{ (\xi, \zeta) \mid \xi \zeta^2 = 1 \} = \{ (\xi, \zeta) \mid \xi = \zeta^{-2} \} = \mathbb{C}^\times$

$Z_1 = \{ x=y=0 \}$
 $Z_2 = \{ u=v=0 \}$

G_k acts on \mathbb{C}^4 with weights $(-2, -2, 1, 3)$

$(\mathbb{C}^4 \setminus Z_1) / \mathbb{C}^\times$ and $(\mathbb{C}^4 \setminus Z_2) / \mathbb{C}^\times$

two quotients, differ by a flip.

Example 2 $P = \mathbb{P}^1 \times \mathbb{P}^1$ weights $\left(\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 0 & -2 \end{array} \right)$

$G_k = \{ (\xi, \zeta) \mid \xi^2 \zeta^2 = 1 \} = \mathbb{C}^\times \times \mathbb{Z}_2$
 $\xi = \pm \zeta^{-1} \quad \zeta = \pm 1$

G_k acts on \mathbb{C}^4 by $(\zeta^{-1}, \zeta^{-1}, \pm \zeta, \pm \zeta)$

$(\mathbb{C}^4 \setminus Z_1) / (\mathbb{C}^\times \times \mathbb{Z}_2)$ and $(\mathbb{C}^4 \setminus Z_2) / (\mathbb{C}^\times \times \mathbb{Z}_2)$

differ by a flop. THE END