Variational Principles for Travelling Waves of Reaction-diffusion Equations

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and
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Reaction-diffusion equations

let $u$ be a vector-valued function, $A$ be a diagonal nonnegative matrix.

$$\frac{\partial u}{\partial t} = A \Delta u + f(u).$$  \hspace{1cm} \text{(RD)}$$

Reaction-diffusion equations describe phenomena in chemical kinetics, physics, and biology such as combustion, phase transitions, propagation of nerve impulses, skin patterns of animals, etc.
Travelling Waves:

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + f(u) \text{ on } \mathbb{R} \\
u(x, t) = u(x - ct)
\end{cases}
\]

Travelling waves usually describe transition process from one equilibrium to another.

Bistable nonlinearity \( f(u) = -u(u - a)(u - 1) \).
Let \( z = x - ct \).

\[
\begin{cases}
  u_t = -cu_z \\
  \Delta_z u + cu_z + f(u) = 0 \text{ on } \mathbb{R}, \\
  \lim_{z \to -\infty} u(z) = 1, \lim_{z \to \infty} u(z) = 0
\end{cases}
\]
Min-max principles for the wave speed

Hadeler and Rothe (1975): \( K = \{ v \in C^2 : v_z < 0, 0 < v < 1 \} \)

\[
\psi(v(z)) = \frac{\Delta v(z) + f(v(z))}{v_z(z)}
\]

Theorem

\[
\sup_{v \in K} \inf_{z \in \mathbb{R}} \psi(v(z)) = c = \inf_{v \in K} \sup_{z \in \mathbb{R}} \psi(v(z)).
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▶ Vol’pert et al. (1994): monotone systems.
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- Assumptions: monotonicity of the wave, uniqueness, stability
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- Assumptions: monotonicity of the wave, uniqueness, stability
- **Question:** Can we use this principle to find the travelling wave solution?
Variational formulation for the solution

c = 0:

Let \( F(u) = -\int_0^u f(s) \, ds \).

\[
E(v) = \int_\mathbb{R} \left[ \frac{v^2}{2} + F(v) \right] \, dz
\]

\[
0 < \eta(z) < 1, \quad \lim_{z \to -\infty} \eta(z) = 1, \quad \lim_{z \to \infty} \eta(z) = 0
\]

\[
S = \{ v = w + \eta : w \in H^1(\mathbb{R}) \}
\]

**Problem:** \( \min_{v \in S} E(v) \)
Variational formulation for the solution

\[ c \neq 0: \quad F(1) < F(0) = 0 \Rightarrow c > 0 \]

\[
E_c(v) = \int_{\mathbb{R}} e^{cz} \left[ \frac{v_z^2}{2} + F(v) \right] \, dz
\]

\[
H_c^1 = \{ v : ||v||_{L^2_c} + ||\nabla v||_{L^2_c} < \infty \}, \quad ||v||_{L^2_c}^2 = \int_{\mathbb{R}} e^{cz} |v|^2 \, dz
\]

**Variational problem:** \( \min_{v \in H_c^1} E_c(v) \)
Variational formulation for the solution

\( c \neq 0: \ F(1) < F(0) = 0 \Rightarrow c > 0 \)

\[
E_c(v) = \int_{\mathbb{R}} e^{cz} \left[ \frac{v_z^2}{2} + F(v) \right] \, dz
\]

\[
H^1_c = \{ v : \| v \|_{L^2_c} + \| \nabla v \|_{L^2_c} < \infty \}, \quad \| v \|_{L^2_c}^2 = \int_{\mathbb{R}} e^{cz} |v|^2 \, dz
\]

Variational problem: \( \min_{v \in H^1_c} E_c(v) \)

Question: Which \( c \) should we choose?
Variational formulation for the solution

c ≠ 0: \( F(1) < F(0) = 0 \Rightarrow c > 0 \)

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E_c(v) = \int_{\mathbb{R}} e^{cz} \left[ \frac{v_z^2}{2} + F(v) \right] dz
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\]

**Variational problem:** \( \min_{v \in H^1_c} E_c(v) \)

▶ Question: Which \( c \) should we choose?

▶ Translation in space:
\( v(z) \) is a travelling wave solution \( \Rightarrow \) So is \( v(z - p) \).
Translation in space:

\[ E_c(v(z - p)) = \int_{\mathbb{R}} e^{cz} \left[ \frac{v_z^2(z - p)}{2} + F(v(z - p)) \right] \, dz \]

\[ = \int_{\mathbb{R}} e^{cz'} + cp \left[ \frac{v_z^2(z')}{2} + F(v(z')) \right] \, dz' = e^{cp} E_c(v(z)). \]

\[
\begin{cases}
\exists E_c(\bar{v}) < 0 \Rightarrow \inf E_c = \lim_{p \to \infty} E_c(\bar{v}(z - p)) = -\infty \\
\forall E_c(v) \geq 0 \Rightarrow \inf E_c = \lim_{p \to -\infty} E_c(\bar{v}(z - p)) = 0
\end{cases}
\]
Variational formulation for the solution

Fife-McLeod (1977): cut-off of $E_c$ to prove stability of travelling wave

**Theorem** (Lucia-Muratov-Novaga, 2008)
Gradient system on a infinite cylinders: $\exists c^*_\ast$ such that $\inf E_{c^*_\ast} = 0$ and it can be attained by a travelling wave solution.

Proof: Constraint

$$\int e^{cz} |\nabla v|^2 = 1.$$

Gallay-Risler (preprint): Use $E_c$ to prove the stability of travelling waves for gradient systems
To kill the translation: \( h(z) = 1 + \tanh \frac{z}{2} \).

\[
E_{h,c}(v) = \int_{\mathbb{R}} h(cz) \left[ \frac{v_z^2}{2} + F(v) \right] dz
\]

\[
S = \{ v = w + \eta : w \in H^1(\mathbb{R}) \}, \quad \lim_{z \to -\infty} \eta(z) = 1, \quad \lim_{z \to \infty} \eta(z) = 0
\]

**Problem:** \( \beta_c = \inf_{v \in S} \sup_{p \in \mathbb{R}} E_{h,c}(v(z - p)) \) a critical value?

\[
v_{zz} + \frac{h_z(cz)}{h(cz)} v_z + f(v) = 0
\]

**Theorem (Chen)**

\( \exists c_1 \) such that:

1. \( \beta_c > 0 \) if \( c > c_1 \); \( \beta_c = 0 \) if \( 0 < c \leq c_1 \),
2. for \( c > c_1 \), \( \exists \) critical point \( v_c \) of \( E_{h,c} \), \( E_{h,c}(v_c) = \beta_c \),
3. \( v_{c_k} \to \) a travelling wave solution \( \bar{v} \).
Gradient system on a infinite cylinders with Neumann boundary condition:

\[
\frac{\partial u}{\partial t} = A \Delta u - \nabla_u F(u).
\]  

(1)

**Theorem**

(1) coercive: \( F(u) \rightarrow \infty \) as \( |u| \rightarrow \infty \);

(2) \( w \) non-degenerate local min. point of \( F \), \( F(w) > \inf F \).

Then \( \exists \) a travelling wave solution \( v \) such that \( v \rightarrow w \) at \( \infty \).
(1) Infinitely many Lyapunov functions

(2) Instantaneous propagation speed
Open Problems

(1) Different diffusion rates
(2) Skew-gradient system