

STRUCTURE OF SOLUTIONS TO CONSERVATION LAWS

YONG JUNG KIM

1. NOTATIONS

Having a better notation is helpful. It helps people to write things simpler and, more importantly, to think things clearer. In the study of systems of conservation laws in a multidimensional space, it seems important to distinguish the effects of the space dimension and of the state variables. In this lecture note we take a notating strategy that reserves row vectors for the vector fields related to the spacial dimension and column ones for the ones related to the state variables. These two kinds of vector fields play different roles in the system and hence one may see that such a simple distinction makes notations systematic.

In our notation all the objects are considered as matrices. We mostly consider real matrices. A real number $r \in \mathbb{R}$ can be considered as a 1×1 matrix. Let n be the space dimension. Then the space variable \mathbf{x} is a vector with n components. Similarly, the velocity vector field \mathbf{u} of a fluid flow in the n -dimensional space and the gradient vector field ∇f of a functional $f : \mathbb{R}^n \rightarrow \mathbb{R}$ are also of such vectors. We consider these space related vector fields as row vector ones, i.e.,

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{u} = (u_1, \dots, u_n), \quad \nabla f = (f_{x_1}, \dots, f_{x_n}) \in \mathbb{R}^{1 \times n}, \quad (1.1)$$

where $\mathbb{R}^{m \times n}$ is the collection of $m \times n$ real matrices. In particular, for the three dimensional space, people typically use $\mathbf{x} = (x, y, z)$ for the space variables and we also use this notation from time to time.

Conservation laws arise from the modeling of physical processes. Consider m -physical quantities, c_1, \dots, c_m . It seems reasonable to write these state variables as a column vector \mathbf{c} due to the horizontal writing of equations:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{R}^{m \times 1}. \quad (1.2)$$

The matrix multiplication is considered as the basic multiplication. Hence, if any other specific notation is not used, a multiplication in this note is the usual matrix multiplication.

Example 1.1 (Taking a gradient $\nabla(\cdot)$ and divergence $\text{div}(\cdot)$). Since the gradient of a real valued functional is a row vector, one obtains a matrix if a gradient of a column vector field is taken, which is actually a Jacobi matrix, i.e.,

$$\nabla \mathbf{c} = \nabla \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} \nabla c_1 \\ \vdots \\ \nabla c_m \end{pmatrix} = \begin{pmatrix} (c_1)_{x_1} & \cdots & (c_1)_{x_n} \\ \vdots & & \vdots \\ (c_m)_{x_1} & \cdots & (c_m)_{x_n} \end{pmatrix} = D\mathbf{c} \in \mathbb{R}^{m \times n}, \quad (1.3)$$

where $D\mathbf{v}$ denotes the Jacobi matrix of the vector field \mathbf{v} . Note that we may apply the operator D to both of a column vector and a row vector field. However, taking a gradient to a row vector field does not give the Jacobi matrix. For example

$$\nabla \mathbf{u} = \nabla(u_1, \dots, u_n) = (\nabla u_1, \dots, \nabla u_n) \in \mathbb{R}^{1 \times n^2}$$

is a longer row vector. If one wants to consider the Jacobi matrix of \mathbf{u} , then it should be written as $D\mathbf{u} = \nabla(\mathbf{u}^t) \in \mathbb{R}^{n \times n}$, where the column vector \mathbf{u}^t is the transpose of the row vector \mathbf{u} .

Now consider taking the divergence of a vector field. It is clear that the number of components of this vector field should be n which is the spacial dimension. Hence we think that the divergence should be applied to a row vector fields. Hence $\text{div}(\mathbf{u})$ and $\text{div}(\nabla f)$ make sense. However, $\text{div}(\mathbf{c})$ does not. (Notice that, even if the number of state variables are same as the dimension, there should be no such thing as $\text{div}(\mathbf{c})$ in a model or in computations.)

One may take a divergence to a matrix if the number of columns of the matrix is n , the space dimension. For example, consider the matrix multiplication $\mathbf{c}\mathbf{u}$. Since they are $m \times 1$ and $1 \times n$ matrices respectively, the matrix multiplication is well defined and the result is a $m \times n$ matrix, i.e.,

$$\mathbf{c}\mathbf{u} = (u_1\mathbf{c}, \dots, u_n\mathbf{c}) = \begin{pmatrix} c_1\mathbf{u} \\ \vdots \\ c_m\mathbf{u} \end{pmatrix} = \begin{pmatrix} c_1u_1 & \cdots & c_1u_n \\ \vdots & & \vdots \\ c_mu_1 & \cdots & c_mu_n \end{pmatrix}.$$

The Jacobi matrix $\nabla \mathbf{c}$ is also $m \times n$ matrix. If the divergence is taken to these matrices, then

$$\text{div}(\mathbf{c}\mathbf{u}) = \text{div} \begin{pmatrix} c_1\mathbf{u} \\ \vdots \\ c_m\mathbf{u} \end{pmatrix} = \begin{pmatrix} \text{div}(c_1\mathbf{u}) \\ \vdots \\ \text{div}(c_m\mathbf{u}) \end{pmatrix}, \quad \text{div}(\nabla \mathbf{c}) = \begin{pmatrix} \text{div}(\nabla c_1) \\ \vdots \\ \text{div}(\nabla c_m) \end{pmatrix} = \Delta \mathbf{c}.$$

Notice that, if both of \mathbf{c} and \mathbf{u} are considered as row or column vectors, then the matrix multiplication $\mathbf{c}\mathbf{u}$ should be replaced with a tensor product such as $\mathbf{c} \otimes \mathbf{u}$. Hence we have saved a tensor product by considering \mathbf{u} as a row vector and \mathbf{c} as a column one. Furthermore the meaning of $\text{div}(\mathbf{c}\mathbf{u})$ is more direct than the one using a tensor product. \square

Example 1.2 (Taking an inner product and a cross product). The inner product, or the dot product, is always between vectors related to the space dimension. The velocity and normal vectors are typical examples. We may generalize it as a product of between two matrices. The requirement is that the rows of each matrix has n -components. In other words the matrices should have n columns. Let A, B be such matrices. Then, the inner product is simply,

$$A \cdot B = AB^t. \quad (1.4)$$

The cross product is only between two vectors in three space dimension. Let $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^{1 \times 3}$. Then the cross produce is defined by

$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2, -(v_1u_3 - v_3u_1), v_1u_2 - v_2u_1). \quad (1.5)$$

It is convenient to consider the operator for the gradient as $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$. Then,

$$\begin{aligned}\nabla c &= \left(\frac{\partial c}{\partial x_1}, \frac{\partial c}{\partial x_2}, \frac{\partial c}{\partial x_3} \right) = \text{grad}(c) \\ \nabla \cdot \mathbf{u} &= \frac{\partial}{\partial x_1} u_1 + \frac{\partial}{\partial x_2} u_2 + \frac{\partial}{\partial x_3} u_3 = \text{div}(\mathbf{u}) \\ \nabla \times \mathbf{u} &= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = \text{curl}(\mathbf{u}). \quad \square\end{aligned}$$

Example 1.3. [Derivation of a convection-reaction-diffusion equations] One can easily model the dynamics of chemical concentrations. Let $c_i = c_i(\mathbf{x}, t)$ be the density of i -th chemical which is a function of space and time variables. Let $S_i(\mathbf{c}, \mathbf{x}, t)$ model the reaction, production and degradation of the i -th chemical c_i . Therefore the vector \mathbf{S} that consists of S_i 's should be considered as a column vector, i.e.,

$$\mathbf{S}(\mathbf{c}, \mathbf{x}, t) = \begin{pmatrix} S_1(\mathbf{c}, \mathbf{x}, t) \\ \vdots \\ S_m(\mathbf{c}, \mathbf{x}, t) \end{pmatrix} \in \mathbb{R}^{m \times 1}. \quad (1.6)$$

Consider a fixed bounded domain Ω in the flow and let \mathbf{n} be the outward unit normal vector at the boundary. The rate of change of the amount of the i -th protein in the region Ω is modelled by

$$\int_{\Omega} \frac{\partial}{\partial t} c_i(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} S_i(\mathbf{c}, \mathbf{x}, t) d\mathbf{x} - \int_{\partial\Omega} \mathbf{j}_i \cdot \mathbf{n} ds, \quad (1.7)$$

where \mathbf{j}_i is the flux of the chemical c_i . This flux consists of two parts. The first one is due to the flux of the fluid flow and the second one is due to the gradient of the concentration. Let τ_i be the corresponding diffusion tensor. Then the flux is modelled by

$$\mathbf{j}_i = c_i \mathbf{u} - \tau_i \nabla c_i. \quad (1.8)$$

After applying the divergence theorem, one obtains

$$(c_i)_t(\mathbf{x}, t) + \text{div}(c_i \mathbf{u}) = \text{div}(\tau_i \nabla c_i) + S_i(\mathbf{c}, \mathbf{x}, t). \quad (1.9)$$

We consider the isotropic case with constant τ_i 's. The the multiplication is actually a scalar multiplication and (1.9) can be rewritten in a vector form, which is

$$\mathbf{c}_t(\mathbf{x}, t) + \text{div}(\mathbf{c}\mathbf{u}) = \tau \Delta \mathbf{c} + \mathbf{S}(\mathbf{c}, \mathbf{x}, t), \quad (1.10)$$

where τ is the diagonal matrix with τ_i in the i -th diagonal element. \square

Remark 1.4. One of the main features of the notation is in the multiplication $\mathbf{c}\mathbf{u}$ in (1.10). Since the space dimension and the number of state variables are different and not one, matrix multiplications such as $\mathbf{u}\mathbf{c}$, $\mathbf{u}\mathbf{u}$ and $\mathbf{c}\mathbf{c}$ are not defined. Hence, such terms should not appear anywhere. Even if the number of state variables is same as the space dimension, such terms are not physical and hence should not appear. However, if one includes tensor product, then tensor products of such combinations are possible. For the sake of remembering formulas or reducing possible mistakes, using a notation that exclude such nonphysical terms is helpful.

Remark 1.5 (Material coordinates and particle trajectories). In this example we will remind that taking the gradient as a row vector makes change of variables convenient. Consider Lagrangian coordinates which are also row vectors and denoted

by $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^{1 \times 3}$. We may implicitly consider the Eulerian coordinates \mathbf{x} as a trajectory function of \mathbf{y} or more explicitly introduce $\Gamma(\mathbf{y}, t)$ which indicates the position of a particle \mathbf{y} at time $t > 0$. In other words we may set

$$\mathbf{x} = \mathbf{x}(\mathbf{y}, t) \quad \text{or} \quad \mathbf{x} = \Gamma(\mathbf{y}, t), \quad \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^{1 \times 3}. \quad (1.11)$$

The Lagrangian coordinate \mathbf{y} actually denotes the particle which was placed at the point \mathbf{y} initially (at $t = 0$) and the trajectory $\Gamma(\mathbf{y}, t)$ denoted the position of the particle at time $t \geq 0$. Therefore, one may set

$$\Gamma(\mathbf{y}, 0) = \mathbf{x}(\mathbf{y}, 0) = \mathbf{y}, \quad \mathbf{u}(\mathbf{y}, t) = \mathbf{u}(\mathbf{x}(\mathbf{y}, t), t) = \frac{\partial \Gamma}{\partial t}(\mathbf{y}, t), \quad (1.12)$$

where \mathbf{u} is the velocity vector field introduced earlier. Note that we abuse our notation here. Using the same notation \mathbf{u} for the velocity, $\mathbf{u}(\mathbf{y}, t)$ denotes the velocity of the particle \mathbf{y} at time $t > 0$ and $\mathbf{u}(\mathbf{x}, t)$ denotes the velocity at the position \mathbf{x} at time $t > 0$. We take the same convention for \mathbf{c} and \mathbf{S} , too.

Remark 1.6 (Material derivative). Consider the acceleration vector $\mathbf{a}(\mathbf{y}, t) = \mathbf{a}(\mathbf{x}(\mathbf{y}, t), t)$. Since the acceleration is the quantity related to the moving particle, the acceleration should be given by

$$\mathbf{a}(\mathbf{y}, t) = \frac{\partial}{\partial t} \mathbf{u}(\mathbf{y}, t) = \frac{\partial}{\partial t} \mathbf{u}(\mathbf{x}(\mathbf{y}, t), t) = \mathbf{u}_t(\mathbf{x}, t) + \nabla \mathbf{u} \cdot \mathbf{u}.$$

This kind of derivative appears whenever a quantity along a trajectory curve is differentiated and is called the material derivative and denoted by $\frac{D}{Dt}(\cdot)$. For example, let $f(\mathbf{y}, t)$ be a certain value that the particle \mathbf{y} has at time $t > 0$. Then, the material derivative is given by

$$\frac{D}{Dt} f(\mathbf{x}, t) = \frac{\partial}{\partial t} f(\mathbf{y}, t) \left(= \frac{\partial}{\partial t} f(\mathbf{x}(\mathbf{y}, t), t) \right) = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) f(\mathbf{x}, t).$$

One can easily check that this material derivative has the properties that a usual differentiation has such as the product rule, linearity and the chain rule. For example

$$\frac{D}{Dt}(f + g) = \frac{D}{Dt}f + \frac{D}{Dt}g, \quad \frac{D}{Dt}(fg) = g \frac{D}{Dt}f + f \frac{D}{Dt}g, \quad \frac{D}{Dt}f(g) = f'(g) \frac{D}{Dt}g.$$

Example 1.7. Let $D_{\mathbf{y}}\Gamma := \Gamma_{\mathbf{y}}$ be the Jacobi matrix of the trajectory, i.e.,

$$\Gamma_{\mathbf{y}} = D_{\mathbf{y}}\Gamma = \nabla_{\mathbf{y}}(\Gamma^t) = \begin{pmatrix} \nabla x_1(\mathbf{y}, t) \\ \nabla x_2(\mathbf{y}, t) \\ \nabla x_3(\mathbf{y}, t) \end{pmatrix} = \begin{pmatrix} (x_1)_{y_1} & (x_1)_{y_2} & (x_1)_{y_3} \\ (x_2)_{y_1} & (x_2)_{y_2} & (x_2)_{y_3} \\ (x_3)_{y_1} & (x_3)_{y_2} & (x_3)_{y_3} \end{pmatrix}.$$

Then, for a vector field F , we obtain the chain rule

$$D_{\mathbf{y}}F = (D_{\mathbf{x}}F)\Gamma_{\mathbf{y}} \quad \text{or} \quad D_{\mathbf{x}}F = (D_{\mathbf{y}}F)\Gamma_{\mathbf{y}}^{-1}. \quad (1.13)$$

Since $DF = \nabla F$ for any column vector field or a scalar function, the chain rule gives

$$\nabla c_i(\mathbf{x}, t) = \nabla c_i(\mathbf{y}, t)\Gamma_{\mathbf{y}}^{-1}, \quad (1.14)$$

$$\text{div}(\mathbf{u}(\mathbf{x}, t)) = \text{Tr}(D\mathbf{u}(\mathbf{x}, t)) = \text{Tr}((D\mathbf{u}(\mathbf{y}, t))\Gamma_{\mathbf{y}}^{-1}), \quad (1.15)$$

$$\Delta c_i(\mathbf{x}, t) = \text{Tr}(D(\nabla c_i(\mathbf{x}, t))) = \text{Tr}(D[\nabla c_i(\mathbf{y}, t)(\Gamma_{\mathbf{y}}^{-1})]\Gamma_{\mathbf{y}}^{-1}), \quad (1.16)$$

where $\text{Tr}(A)$ is the trace of a square matrix A . Notice that, if the gradient is considered as a column vector, one should take a transpose of a gradient vector and, hence, the notation becomes messy, which is undesirable. (Independent variables of

the differential operators in the previous calculations should be clear. For example in the writing $\nabla c_i(\mathbf{x}, t)$ the independent variables are \mathbf{x} and t and hence the gradient operator is with respect to \mathbf{x} . On the other hand, specifying independent variables of each term makes lengthy equations. Hence we specify only one of them for each side of equalities.)

Similar change of variables with respect to the time variable $t > 0$, we obtain

$$\mathbf{c}_t(\mathbf{x}, t) + \text{div}(\mathbf{c}\mathbf{u}) = \mathbf{c}_t(\mathbf{x}, t) + (\nabla\mathbf{c})\mathbf{u}^t + \mathbf{c}\text{div}\mathbf{u} = \mathbf{c}_t(\mathbf{y}, t) + \mathbf{c}\text{Tr}((D\mathbf{u})\Gamma_{\mathbf{y}}^{-1}). \quad (1.17)$$

Therefore, (1.10) is transformed to

$$(c_i)_t(\mathbf{y}, t) = \tau_i \text{Tr}(D[(\nabla c_i)\Gamma_{\mathbf{y}}^{-1}]\Gamma_{\mathbf{y}}^{-1}) + S_i(\mathbf{c}, \mathbf{y}, t) - c_i \text{Tr}((D\mathbf{u})\Gamma_{\mathbf{y}}^{-1}), \quad i = 1, 2, 3. \quad (1.18)$$

Notice that the equation (1.18) is written in a scalar form since taking the Jacobian of $\nabla\mathbf{c}$ is not clearly defined. \square

Remark 1.8. Consider the products between $\nabla\mathbf{c}$ and \mathbf{u} or their transposes. Then there are eight possibilities. Since \mathbf{u} is a row vector, the products such as $(\nabla\mathbf{c})\mathbf{u}$, $(\nabla\mathbf{c})^t\mathbf{u}$, $\mathbf{u}^t(\nabla\mathbf{c})$ and $\mathbf{u}^t(\nabla\mathbf{c})^t$ are not defined. Furthermore, $\mathbf{u}(\nabla\mathbf{c})$ and $\mathbf{u}(\nabla\mathbf{c})^t$ are defined only if the number of state variables is same as the space dimension, which is not of our case. Hence those six products should not appear in any case. Notice that, even if the dimension and the number of state variables are same, a product such as $\mathbf{u}(\nabla\mathbf{c})$ and $\mathbf{u}^t(\nabla\mathbf{c})^t$ has no physical meaning. The other two are $(\nabla\mathbf{c})\mathbf{u}^t$ and $\mathbf{u}(\nabla\mathbf{c})^t$, which are column and row vectors, respectively, and have identical components. Hence $(\nabla\mathbf{c})\mathbf{u}^t$ is the only possible case that may appear.

Remark 1.9. Notations of this note are like followings.

- (1) Bold characters are vectors.
- (2) \mathbf{v}^t : transpose of the vector \mathbf{v} .
- (3) \mathbf{u} : velocity row vector.
- (4) \mathbf{c} : state column vector.
- (5) $\nabla\mathbf{c}$: gradient row vector. $\nabla\mathbf{c}$: Jacobi matrix of the vector field \mathbf{c} .
- (6) τ : (diagonal) diffusion matrix.
- (7) $D\mathbf{f}$: Jacobi matrix of a vector field \mathbf{f} . If \mathbf{f} is a column vector, then $D\mathbf{f} = \nabla\mathbf{f}$.
If \mathbf{f} is a row vector, then $D\mathbf{f} = \nabla(\mathbf{f}^t)$.
- (8) \tilde{u} is a column velocity and $\nabla^t f = (\nabla f)^t$ is a column gradient vector.

2. THE EQUATIONS OF MOTION

In this section we consider equations motion given by the conservation laws. From now on we consider three dimensional space only and hence $\mathbf{x} \in \mathbb{R}^{1 \times 3}$.

2.1. Thermodynamics. In thermodynamics the following five quantities are basic state quantities which are functions of (\mathbf{x}, t) depending on the given flow:

- ρ : mass density,
- e : specific internal energy density,
- s : specific entropy density,
- p : pressure,
- T : temperature.

Three of these five are called density functions since the corresponding quantities, mass, energy and entropy, are not local properties. In fact, they actually have meaning when they are integrated over a domain. For example the mass of the fluid at a given point is meaningless. However, the mass of the fluid in a regions Ω does have meaning and it is given by

$$\int_{\Omega} \rho d\mathbf{x} = \text{Mass of the fluid in a region } \Omega.$$

The other two densities are specific ones or, in other words, densities per unit of mass. This means that the actual density is the one after multiplying the mass density, i.e.,

$$\int_{\Omega} \rho e d\mathbf{x} = \text{Internal energy of the fluid in a region } \Omega,$$

$$\int_{\Omega} \rho s d\mathbf{x} = \text{Entropy of the fluid in a region } \Omega.$$

Remark 2.1 (a nonsense quiz). What is the specific mass density? It is 1.

On the other hand the other two quantities, the pressure and the temperature, are local ones. Hence, the temperature or the pressure at a given point is meaningful. In the followings we simply call ρ, e and s as mass, internal energy and entropy for brevity. It is assumed that, if two of the five physical quantities are given, then the other three are decided by the laws of thermodynamics. We consider the mass ρ and the pressure p are unknowns and the other three are given by these two.

The following three quantities play important role in the evolution of the flow:

$$V = 1/\rho : \text{(specific) volume,}$$

$$h = e + p/\rho = e + pV : \text{(specific) enthalpy density,}$$

$$E = e + \frac{1}{2}|\mathbf{u}|^2 : \text{(specific) total energy density.}$$

Since E is the specific total energy density, the total energy of the fluid is given by

$$\int_{\Omega} (\rho E) d\mathbf{x} = \text{Total energy of the fluid in a region } \Omega.$$

It is clear why V is the specific volume since

$$\int_{\Omega} (\rho V) d\mathbf{x} = \int_{\Omega} 1 d\mathbf{x}$$

is the volume of the domain Ω .

Remark 2.2. It is not clear yet how the other three quantities, s, p, T , are given from ρ, p . In Majda, it is said that the internal energy is given first, i.e., $e = e(\rho, p)$, and then, the others two are given by the second law of thermodynamics,

$$Tds = de - \frac{p}{\rho^2}d\rho = de + dV. \quad (2.1)$$

This law can be written in terms of enthalpy h which is

$$Tds = dh - \frac{1}{\rho}dp = dh - Vdp. \quad (2.2)$$

Remark 2.3 (Ideal Gases). (i) For the ideal gases, the other three are given by

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)} = \frac{T}{\gamma - 1}, \quad (2.3)$$

$$T(\rho, p) = \frac{p}{\rho}, \quad (2.4)$$

$$e^s = p\rho^{-\gamma}, \quad \gamma > 1, \text{ constant.} \quad (2.5)$$

(ii) The power law is often used for the pressure of isentropic gases given by

$$p = p(\rho) = A\rho^\gamma \quad \text{with constants } \gamma \geq 1 \text{ and } A > 0.$$

Then, the internal energy and the enthalpy are

$$e = \frac{A\rho^{\gamma-1}}{\gamma-1} \quad \text{and} \quad h = \gamma e = \frac{\gamma A\rho^{\gamma-1}}{\gamma-1}.$$

2.2. Euler's Equation. In this section we derive the Euler's equations that give the motion of a compressible flow. The derivation is based on the conservation or the balance of physical quantities.

The mass, the momentum and the total energy are conserved quantities. Note that the velocity \mathbf{u} is a row vector. Hence, if we set the momentum as $\rho\mathbf{u}$, it is a row vector. However, we want to set the state variables as a column vector. Hence we introduce a column velocity vector \tilde{u} by taking the transpose of \mathbf{u} and the state vector

$$\mathbf{c} := \begin{pmatrix} \rho \\ \rho\tilde{u} \\ \rho E \end{pmatrix} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho u_3 \\ \rho E \end{pmatrix} \in \mathbb{R}^{5 \times 1}. \quad (2.6)$$

The derivation of the Euler equation is essentially same as the one for the convection-reaction-diffusion equation in Example 1.3. The difference is that Euler equations are for the inviscid case and hence there is no diffusion.

Let $S_i(\mathbf{c}, \mathbf{x}, t)$ model the source or sink of the i -th quantity c_i and \mathbf{S} the corresponding column source vector. Consider a fixed bounded domain Ω in the flow and let \mathbf{n} be the outward unit normal vector at the boundary. The rate of change of the amount of the state variables in the region Ω is modelled by

$$\frac{\partial}{\partial t} \int_{\Omega} \mathbf{c}(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} \mathbf{S}(\mathbf{c}, \mathbf{x}, t) d\mathbf{x} - \int_{\partial\Omega} \mathbf{c}\mathbf{u} \cdot \mathbf{n} ds,$$

where the flux of the state variables is simply $\mathbf{c}\mathbf{u}$. Notice that the flux for the i -th quantity is $c_i\mathbf{u}$ which is same as the one in (1.8) with zero diffusion tensor $\tau_i = 0$. Hence, one obtains

$$\mathbf{c}_t(\mathbf{x}, t) + \operatorname{div}(\mathbf{c}\mathbf{u}) = \mathbf{S}(\mathbf{c}, \mathbf{x}, t). \quad (2.7)$$

The mass is not produced nor disappeared and hence we set $S_1 = 0$. Hence the equation for the mass conservation is given by

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0. \quad (2.8)$$

The following transport theorem depends on this mass conservation law. Let Ω_0 be a domain in a Lagrangian space. In other words one may consider Ω_0 as a collection of particles or a domain of a flow at the initial time $t = 0$. Then, let Ω_t be the domain in the Eulerian space that the particles in Ω_0 occupies, i.e.,

$$\Omega_t = \{\Gamma(\mathbf{y}, t); \mathbf{y} \in \Omega_0\}.$$

Theorem 2.4 (Transport Theorem).

$$\frac{d}{dt} \int_{\Omega_t} f(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_t} \left(\frac{D}{Dt} f + f \operatorname{div} \mathbf{u} \right) d\mathbf{x} = \int_{\Omega_t} \left(\frac{\partial}{\partial t} f + \operatorname{div}(f\mathbf{u}) \right) d\mathbf{x}. \quad (2.9)$$

$$\frac{d}{dt} \int_{\Omega_t} \rho f(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_t} \rho \frac{D}{Dt} f(\mathbf{x}, t) d\mathbf{x}. \quad (2.10)$$

Proof. First consider

$$\frac{d}{dt} \int_{\Omega_t} f(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega_0} \frac{d}{dt} \left(f(\mathbf{x}(\mathbf{y}, t), t) J(\mathbf{y}, t) \right) d\mathbf{y} = \int_{\Omega_0} \left(\frac{d}{dt} (f) J + f \frac{\partial}{\partial t} J \right) d\mathbf{y},$$

where $J = |D_{\mathbf{y}}\mathbf{x}| = |\nabla_{\mathbf{y}}\mathbf{x}^t|$ is the volume factor for the change of variables. Then, due to the multi-linear property of the determinant, we have

$$\frac{\partial}{\partial t} J = \frac{\partial}{\partial t} \begin{vmatrix} \nabla x_1(\mathbf{y}, t) \\ \nabla x_2(\mathbf{y}, t) \\ \nabla x_3(\mathbf{y}, t) \end{vmatrix} = \begin{vmatrix} \nabla u_1(\mathbf{y}, t) \\ \nabla x_2(\mathbf{y}, t) \\ \nabla x_3(\mathbf{y}, t) \end{vmatrix} + \begin{vmatrix} \nabla x_1(\mathbf{y}, t) \\ \nabla u_2(\mathbf{y}, t) \\ \nabla x_3(\mathbf{y}, t) \end{vmatrix} + \begin{vmatrix} \nabla x_1(\mathbf{y}, t) \\ \nabla x_2(\mathbf{y}, t) \\ \nabla u_3(\mathbf{y}, t) \end{vmatrix}.$$

The chain rule gives

$$D_{\mathbf{y}}\mathbf{u}(\mathbf{x}(\mathbf{y}, t), t) = (D_{\mathbf{x}}\mathbf{u})(D_{\mathbf{y}}\mathbf{x}).$$

Hence one can easily see that $\nabla_{\mathbf{y}}u_i(\mathbf{x}(\mathbf{y}, t), t) = \sum_{k=1}^3 (u_i)_{x_k} \nabla x_k$. Hence, substituting these into the previous equation gives

$$\frac{\partial}{\partial t} J = (u_1)_{x_1} J + (u_2)_{x_2} J + (u_3)_{x_3} J = (\operatorname{div} \mathbf{u}) J. \quad (2.11)$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} f(\mathbf{x}, t) d\mathbf{x} &= \int_{\Omega_0} \left(\frac{d}{dt} f + f \operatorname{div} \mathbf{u} \right) J d\mathbf{y} = \int_{\Omega_t} \left(\frac{D}{Dt} f + f \operatorname{div} \mathbf{u} \right) d\mathbf{x} \\ &= \int_{\Omega_t} \left(\frac{\partial}{\partial t} f + \operatorname{div}(f\mathbf{u}) \right) d\mathbf{x}. \end{aligned}$$

Therefore, (2.9) holds. Now replace f in (2.9) with ρf . Then, since

$$\begin{aligned} \frac{D}{Dt}(\rho f) + (\rho \operatorname{div} \mathbf{u}) f &= \rho \frac{D}{Dt} f + f \frac{\partial}{\partial t} \rho + f \nabla \rho \cdot \mathbf{u} + f(\rho \operatorname{div} \mathbf{u}) \\ &= \rho \frac{D}{Dt} f + f \left(\frac{\partial}{\partial t} \rho + \operatorname{div}(\rho \mathbf{u}) \right) = \rho \frac{D}{Dt} f, \end{aligned}$$

the equation (2.11) holds. \square

2.2.1. *Balance of Momentum.* Now consider the momentum conservation. Using the transport theorem, one can easily check that the rate of change of the momentum is actually the force applied to the flow:

$$\frac{d}{dt} \int_{\Omega_t} \rho \tilde{u} d\mathbf{x} = \int_{\Omega_t} \rho \frac{D}{Dt} \tilde{u} d\mathbf{x}.$$

Hence, to obtain the momentum conservation law, we should consider the force. The first one is the body force such as the gravity, which can be written as

$$\int_{\Omega_t} \rho \mathbf{b} d\mathbf{x} : \text{external body force.}$$

Note that we are using the notation \mathbf{b} for the external body force. Since the force is a vector in a space, it is considered as a row vector in our notational convention. However, in the momentum conservation law, since the velocity is a state variable, this body force should be in a form of a column vector. Therefore, we put \mathbf{b}^t .

The pressure also plays as a force as in the following mechanism. First we assume that the fluid is ideal in the sense that the pressure give the surface force which is orthogonal to the surface. Then the surface force induced by the pressure applied to the domain Ω_t is given by

$$- \int_{\partial\Omega_t} p(\mathbf{x}, t) \mathbf{n}^t dA : \text{surface force,}$$

where dA indicates the surface integration and \mathbf{n} is the outward unit normal vector to the surface.

can be increased due to the external body force such as the gravity and surface force such as the pressure. Notice that, since \mathbf{n} is the outward normal vector, there is a negative sign. If not, the force is the one contributed to the outside of the region. We can apply the divergence theorem and obtain

$$\int_{\partial\Omega_t} p \mathbf{n}^t dA = \int_{\partial\Omega_t} \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} \cdot \mathbf{n} dA = \int_{\Omega_t} \begin{pmatrix} \text{div}(p \ 0 \ 0) \\ \text{div}(0 \ p \ 0) \\ \text{div}(0 \ 0 \ p) \end{pmatrix} d\mathbf{x} = \int_{\Omega_t} \nabla^t p d\mathbf{x},$$

where $\nabla^t p$ is a column gradient vector, i.e.,

$$\nabla^t p = (\nabla p)^t.$$

Finally, combining these three terms one obtains the momentum conservation law:

$$\rho \frac{D}{Dt} \tilde{u} = -\nabla^t p + \rho \mathbf{b}^t. \quad (2.12)$$

Furthermore, since $\rho_t + \text{div}(\rho \mathbf{u}) = 0$, we have

$$\rho \frac{D}{Dt} \tilde{u} = \rho \tilde{u}_t + \rho \nabla \tilde{u} \cdot \mathbf{u} + (\rho_t + \text{div}(\rho \mathbf{u})) \tilde{u} = (\rho \tilde{u})_t + \text{div}(\rho \tilde{u} \mathbf{u}).$$

Hence (2.12) can be rewritten as

$$(\rho \tilde{u})_t + \text{div}(\rho \tilde{u} \mathbf{u}) = -\nabla^t p + \rho \mathbf{b}^t, \quad (2.13)$$

which fits in the form (2.7).

2.2.2. *Balance of Energy.* Now consider the balance of energy. The total energy is the sum of the kinetic energy and the internal energy,

$$E = e + \frac{1}{2}|\mathbf{u}|^2 = e + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}.$$

The change of the kinetic energy of certain portion Ω_t of the fluid should be same as the work done to the portion by the surface force induced by the pressure or the external body force. Note that the work is given by ‘work = distance \times force’, where the distance is actually the component of the force vector. Hence its rate of change is given by

$$\frac{d}{dt}\text{work} = \text{velocity} \cdot \text{force}.$$

Suppose that the change of the total energy of the flow is caused by the work done by the pressure surface force and the body force. Then, it is written as

$$\frac{d}{dt} \int_{\Omega_t} \rho E(\mathbf{x}, t) dx = - \int_{\partial\Omega_t} \mathbf{u} \cdot p \mathbf{n} dA + \int_{\Omega_t} \mathbf{u} \cdot \rho \mathbf{b} dx.$$

Then, the transport and the divergence theorem gives

$$\int_{\Omega_t} \rho \frac{D}{Dt} E(\mathbf{x}, t) dx = - \int_{\Omega_t} \text{div}(p\mathbf{u}) d\mathbf{x} + \int_{\Omega_t} \mathbf{u} \cdot \rho \mathbf{b} dx.$$

Hence the conservation of energy is written as

$$\rho \frac{D}{Dt} E + \text{div}(p\mathbf{u}) = \rho \mathbf{u} \cdot \mathbf{b}. \quad (2.14)$$

Furthermore, since

$$(\rho E)_t + \text{div}(\rho E \mathbf{u}) = (\rho_t + \text{div}(\rho \mathbf{u}))E + \rho(E_t + \mathbf{u} \cdot \nabla E) = \rho \frac{D}{Dt} E,$$

(2.14) can be written as

$$(\rho E)_t + \text{div}((\rho E + p)\mathbf{u}) = \rho \mathbf{u} \cdot \mathbf{b}. \quad (2.15)$$

Notice that this energy balance law is based on the assumption that the internal energy is constant.

If the Euler’s equations are written in the conservative form (2.7), then the corresponding state variables and the source term are

$$\mathbf{c} := \begin{pmatrix} \rho \\ \rho \tilde{u} \\ \rho E \end{pmatrix}, \quad \mathbf{S} := \begin{pmatrix} 0 \\ -\nabla^t p + \rho \mathbf{b}^t \\ -\text{div}(p\mathbf{u}) + \rho \mathbf{u} \cdot \mathbf{b} \end{pmatrix} \in \mathbb{R}^{5 \times 1}. \quad (2.16)$$

Then the Euler equations are written as

$$\mathbf{c}_t + \text{div}(\mathbf{c}\mathbf{u}) = \mathbf{S}(\rho, p, \nabla p, \mathbf{u}, \mathbf{b}), \quad (2.17)$$

which is in the same form as (1.10) with a different reaction part \mathbf{S} and zero viscosity $\tau = 0$. Hence the structure that characterizes the Euler equation is in the reaction part $\mathbf{S} = \mathbf{S}(\rho, p, \nabla p, \mathbf{u}, \mathbf{b})$ given by the pressure and the external force field.

2.3. Special cases.

2.3.1. *Incompressible flow with constant internal energy.* A flow is called incompressible if $\operatorname{div} \mathbf{u} = 0$. Then, since $\frac{d}{dt} J = \operatorname{div} \mathbf{u} J = 0$ and $J(0) = |I| = 1$, incompressibility is equivalent to $J = 1$. One can easily see that the volume of the fluid is not changed if $\operatorname{div} \mathbf{u} = 0$. Consider

$$\frac{d}{dt} \int_{\Omega_t} 1 d\mathbf{x} = \int_{\Omega_0} \frac{d}{dt} J d\mathbf{x} = \int_{\Omega_t} \operatorname{div} \mathbf{u} J d\mathbf{x} = 0.$$

Hence, the volume which is occupied by the particles in Ω_0 is constant if $\operatorname{div} \mathbf{u} = 0$.

The material derivative of the kinetic energy is

$$\frac{1}{2} \frac{D}{Dt} (\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2} \left(\frac{D}{Dt} \mathbf{u} \right) \cdot \mathbf{u} + \frac{1}{2} \mathbf{u} \cdot \left(\frac{D}{Dt} \mathbf{u} \right) = \left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \mathbf{u}.$$

Hence, if the internal energy is assumed to be constant, then

$$\rho \frac{D}{Dt} E + \operatorname{div}(p\mathbf{u}) = \rho \left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \mathbf{u} + \nabla p \cdot \mathbf{u} + p \operatorname{div} \mathbf{u}.$$

Since the flow is incompressible, $\operatorname{div} \mathbf{u} = 0$, the balance of energy is written as

$$\rho \left(\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) \cdot \mathbf{u} = (-\nabla p + \mathbf{b}) \cdot \mathbf{u},$$

which simply comes from the balance of momentum. Therefore, if flow is incompressible and the internal energy is invariant, then the energy equation is dropped and the Euler's equations is written as

$$\frac{D\rho}{Dt} = 0 \tag{2.18}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla^t p + \rho \mathbf{b}^t \tag{2.19}$$

$$\operatorname{div} \mathbf{u} = 0 \tag{2.20}$$

with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial D. \tag{2.21}$$

Remark 2.5. It is not surprising to get the energy conservation dropped. According to the hypothesis of thermodynamics the five basic quantities are decided if two of them are given. Hence, if the internal energy 'e' is constant then all the others are functions of one of them, say ρ . Then there are four unknowns, \mathbf{u} and ρ , and, hence, only four equations are needed to close the system. However, there are five equations above. Note that the last equation $\operatorname{div} \mathbf{u} = 0$ actually plays a role of an assisting relation.

2.3.2. *Compressible flow with constant entropy.* Suppose that the entropy is constant. Then, (2.1) and (2.2) are written as

$$dh = \frac{1}{\rho} dp, \quad de = \frac{p}{\rho^2} d\rho,$$

and one may easily obtain the enthalpy and the internal energy, i.e.,

$$h = \int^p \frac{p'(k)}{k} dk, \quad e = \int^p \frac{p(k)}{k^2} dk, \quad p = \rho^2 \frac{\partial e}{\partial \rho}.$$

The material derivative of the internal energy $e = e(\rho)$ is given by

$$\frac{D}{Dt} e(\rho) = e'(\rho) \frac{D}{Dt} \rho = -\rho e'(\rho) \operatorname{div} \mathbf{u}.$$

Using the relation, the balance of energy is written as

$$\begin{aligned} 0 &= \rho \frac{DE}{Dt} + \operatorname{div}(p\mathbf{u}) - \rho\mathbf{u} \cdot \mathbf{b} = \rho \left(\frac{D\mathbf{u}}{Dt} \cdot \mathbf{u} - \rho e'(\rho) \operatorname{div}\mathbf{u} \right) + p \operatorname{div}\mathbf{u} + \nabla p \cdot \mathbf{u} - \rho\mathbf{u} \cdot \mathbf{b} \\ &= \left(\rho \frac{D\mathbf{u}}{Dt} + \nabla p - \rho\mathbf{b} \right) \cdot \mathbf{u}, \end{aligned}$$

which comes from the balance of the momentum and hence can be dropped. Since $\nabla p = \rho \nabla h$, the Euler's equations is written as

$$\frac{D\rho}{Dt} + \rho \operatorname{div}\mathbf{u} = 0 \quad (2.22)$$

$$\frac{D\tilde{\mathbf{u}}}{Dt} = -\nabla^t h + \mathbf{b}^t \quad (2.23)$$

with the boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial D. \quad (2.24)$$

2.3.3. *A non-conservative form and the shallow water equation.* The pressure of the ideal isentropic gas is given by $p = A\rho^\gamma$, $\gamma \geq 1$. Then,

$$\frac{Dp}{Dt} = p'(\rho) \frac{D\rho}{Dt} = \gamma A \rho^{\gamma-1} \frac{D\rho}{Dt}$$

and the balance of mass becomes

$$\frac{Dp}{Dt} + \gamma p \operatorname{div}\mathbf{u} = \gamma A \rho^{\gamma-1} \frac{D\rho}{Dt} + \gamma A \rho^\gamma \operatorname{div}\mathbf{u} = \gamma A \rho^{\gamma-1} \left(\frac{D\rho}{Dt} + \rho \operatorname{div}\mathbf{u} \right) = 0.$$

Therefore, the motion of the isentropic gas is written as

$$\frac{Dp}{Dt} + \gamma p \operatorname{div}\mathbf{u} = 0, \quad (2.25)$$

$$\rho \frac{D\tilde{\mathbf{u}}}{Dt} = -\nabla^t p + \mathbf{b}^t, \quad (2.26)$$

where the entropy is assumed to be constant $s(\mathbf{x}, t) = s_0$. The relation between A and s_0 directly comes from (2.5) and $A = e^{s_0}$. Note that this equation is not in a conservative form and hence agrees with the original problem in the smooth regions. This system is also called shallow water equations with $n = 1$ or 2 and $\gamma = 2$.

2.3.4. *The nonlinear wave equation.* The nonlinear wave equation is given by

$$u_{tt} = (k(u_x))_x.$$

Let $u_1 = u_x$ and $u_2 = u_t$. Then,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} u_2 \\ k(u_1) \end{pmatrix}_x.$$

In fluid dynamics the one dimensional isentropic gas in Lagrangian coordinates is given in this form, where u_1 is the specific volume and u_2 is the velocity and

$$k(u) = -Au^{-\gamma}, \quad A > 0, \gamma > 1.$$

2.4. Bernoulli's theorem. One may consider the curve traced out by a particle which is called a **trajectory**. The trajectory $\mathbf{x} = \mathbf{x}(t)$ is given by

$$\mathbf{x}'(t) = \mathbf{u}(\mathbf{x}(t), t).$$

However, the **streamline** $\mathbf{x} = \mathbf{x}(t)$ is an integral curve of the velocity vector field at the given time $t \geq 0$, i.e.,

$$\mathbf{x}'(\tau) = \mathbf{u}(\mathbf{x}(\tau), t).$$

The flow is called stationary if the velocity vector is not changed in time, i.e., $\mathbf{u}_t = 0$. Then, $\mathbf{u}(\mathbf{x}(\tau), t) = \mathbf{u}(\mathbf{x}(\tau), \tau)$ for all τ and t and hence the streamlines and the particle trajectories are identical. The Bernoulli's theorem is about such a flow.

Theorem 2.6 (Bernoulli's). *Consider a stationary flow. (i) In an isentropic flow with a potential body force $\mathbf{b} = -\nabla\phi$ the quantity $\frac{1}{2}\|\mathbf{u}\|^2 + h + \phi$ is constant along a streamline. (ii) If the flow is homogeneous ($\rho = \rho(t)$ constant in space) and incompressible (hence, $\rho = \rho(0)$ constant in time, too) and a potential body force $\mathbf{b} = -\nabla\phi$ is given, then $\frac{1}{2}\|\mathbf{u}\|^2 + p/\rho(0) + \phi$ is constant along a streamline.*

Proof. (i) Since the flow is stationary, one may set $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{u}(\mathbf{x})\|^2 + (h + \phi)(\mathbf{x})$ and Eq. (2.23) is written as

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla(h + \phi).$$

Using the identity

$$\nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) = (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}),$$

one obtains

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla\left(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) - (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla\left(\frac{1}{2}\|\mathbf{u}\|^2 + h + \phi\right) = \nabla f.$$

Hence, for a given streamline $\mathbf{x}(t)$, one obtains

$$f(\mathbf{x}(t)) - f(\mathbf{x}(0)) = \int_0^t \nabla f(\mathbf{x}(\tau)) \cdot \mathbf{x}'(\tau) d\tau = \int_0^t \mathbf{u} \times (\nabla \times \mathbf{u}) \cdot \mathbf{u}(\mathbf{x}(\tau)) d\tau = 0.$$

For the case (ii), the proof is clear since Eq. (2.19) is written as

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\left(\frac{p}{\rho(0)} + \phi\right).$$

□

3. ROTATION AND VORTICITY

In this section we consider the rotation and the deformation of a flow. The curl of a velocity vector field is called the vorticity and we denote it by ξ , i.e.,

$$\xi = \nabla \times \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^{1 \times 3}.$$

3.1. decomposition of velocity. Let $\mathbf{h} \in \mathbb{R}^{1 \times 3}$ be small. Then the Taylor's expansion theorem gives that

$$\tilde{u}(\mathbf{x} + \mathbf{h}) = \tilde{u}(\mathbf{x}) + (\nabla \tilde{u})\mathbf{h}^t + O(|\mathbf{h}|^2),$$

where $\nabla \tilde{u}$ is the Jacobi matrix. Let

$$D = \frac{1}{2}(\nabla \tilde{u} + (\nabla \tilde{u})^t), \quad S = \frac{1}{2}(\nabla \tilde{u} - (\nabla \tilde{u})^t).$$

Then $\nabla \tilde{u} = D + S$ and

$$S = \frac{1}{2} \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Hence one can easily see that

$$S \cdot \mathbf{h} = S\mathbf{h}^t = \frac{1}{2}(\xi \times \mathbf{h})^t,$$

and, if $\nabla \tilde{u}$ is symmetric, then S is the null matrix. Therefore, the Taylor expansion becomes

$$\mathbf{u}(\mathbf{x} + \mathbf{h}) = \mathbf{u}(\mathbf{x}) + \mathbf{h}D(\mathbf{x}) + \frac{1}{2}\xi(\mathbf{x}) \times \mathbf{h} + O(|\mathbf{h}|^2), \quad (3.1)$$

where ξ is the vorticity and $D(x)$ is a symmetric tensor. Notice that the terms in Eq. (3.1) are in row vectors. The term $\mathbf{h}D(\mathbf{x})$ is due to the symmetry of $D(\mathbf{x})$, i.e.,

$$(D(\mathbf{x})\mathbf{h}^t)^t = \mathbf{h}D(\mathbf{x})^t = \mathbf{h}D(\mathbf{x}).$$

We want to analyze the local effect of the velocity vector field and after a translation we may consider the case $\mathbf{x} = 0$ only. Then (3.1) is written as

$$\mathbf{u}(\mathbf{h}) = \mathbf{u}(0) + \mathbf{h}D(0) + \frac{1}{2}\xi(0) \times \mathbf{h} + O(|\mathbf{h}|^2). \quad (3.2)$$

The velocity \mathbf{u} has been now decomposed into three parts near the origin. A simple way to analyze their effects is a splitting approach, which is simply consider the effects of a single term treating as if there is no other terms. The over all effects can be understood summing up all of them.

3.1.1. transportation. Let $\mathbf{h}(t)$ be a trajectory of a particle and hence $\mathbf{h}'(t) = \mathbf{u}(\mathbf{h}(t))$. We will see how does the velocity vector field effects the particle trajectory. The effect of the first part is very simple. Since $\mathbf{h}'(t) = \mathbf{u}(\mathbf{h}(t)) = \mathbf{u}(0)$, we have

$$\mathbf{h}(t) = t\mathbf{u}(0) + \mathbf{h}(0),$$

which is a simple transformation with a constant velocity. Hence it does not give any relative change of the flow.

3.1.2. *deformation tensor.* Consider the role of the symmetric tensor $D(\mathbf{x})$ in (3.1). Because D is a symmetric matrix,

$$\mathbf{h}D(0) = \nabla\psi(\mathbf{h}), \quad \psi(\mathbf{h}) = \frac{1}{2}\mathbf{h}D(0)\mathbf{h}^t.$$

Since $D(0)$ is real symmetric, there exist eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and hence the matrix representation of $D(0)$ is

$$D(0) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

using $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as its basis. Hence the relation $\mathbf{h}'(t) = \mathbf{h}D(0)$ gives

$$h'_i(t) = \lambda_i h_i(t), \quad i = 1, 2, 3.$$

Therefore,

$$h_i(t) = h_i(0)e^{\lambda_i t}, \quad i = 1, 2, 3.$$

Hence the volume near the origin is

$$V(t) = h_1(t)h_2(t)h_3(t) = h_1(0)h_2(0)h_3(0)e^{(\lambda_1 + \lambda_2 + \lambda_3)t}$$

Hence the rate of the change of the volume is

$$V'(t) = (\lambda_1 + \lambda_2 + \lambda_3)V(t) = \text{Trace}(D(0))V(t) = (\text{div}\mathbf{u}(0))V(t).$$

3.1.3. *rotation.* Consider the third term

$$\mathbf{h}'(t) = \frac{1}{2}\xi(0) \times \mathbf{h}(t).$$

It is clear that the velocity vector is normal to the plane spanned by two vectors $\xi(0)$ and $\mathbf{h}(t)$ with magnitude $\frac{1}{2}|\xi(0)||\mathbf{h}|\sin(\theta)$ where θ is the angle between $\xi(0)$ and \mathbf{h} . Hence the particle trajectory $\mathbf{h}(t)$ gives a circular motion around the vorticity vector $\xi(0)$ with speed $\frac{1}{2}|\xi(0)||\mathbf{h}|\sin(\theta)$. (It takes your imagination to see the picture.) One can find a rotation matrix

$$\mathbf{h}(t) = M\left(\frac{1}{2}|\xi(0)|t, \xi(0)\right)\mathbf{h}(0),$$

where $M\left(\frac{1}{2}|\xi(0)|t, \xi(0)\right)$ is the matrix that represents a rotation through an angle $\frac{1}{2}|\xi(0)|t$ about the axis $\xi(0)$.

3.2. **circulation.** Let C be a simple closed contour in a flow at time $t = 0$. Let $C_t = \Gamma(C, t)$ which is the contour carried by the flow after time $t > 0$. The circulation around C_t is defined by the line integral of the velocity field, i.e.,

$$\Gamma_{C_t} = \oint_{C_t} \mathbf{u} \cdot ds.$$

Lemma 3.1.

$$\frac{d}{dt} \oint_{C_t} \mathbf{u} \cdot ds = \oint_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds$$

Proof. Let $\mathbf{y}(s)$, $0 < s < 1$, be a parametrization of the contour C . Then,

$$\begin{aligned} \frac{d}{dt} \oint_{C_t} \mathbf{u} \cdot ds &= \frac{d}{dt} \int_0^1 \mathbf{u}(\Gamma(\mathbf{x}(s), t), t) \cdot \frac{d}{ds} \Gamma(\mathbf{y}(s), t) ds \\ &= \int_0^1 \frac{D\mathbf{u}}{Dt} \cdot \frac{\partial}{\partial s} \Gamma(\mathbf{y}(s), t) ds + \int_0^1 \mathbf{u} \cdot \frac{\partial}{\partial s} \mathbf{u}(\Gamma(\mathbf{y}(s), t), t) ds \\ &= \oint_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds + \int_0^1 \frac{1}{2} \frac{\partial}{\partial s} (\mathbf{u} \cdot \mathbf{u})(\Gamma(\mathbf{y}(s), t), t) ds = \oint_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds. \end{aligned}$$

□

Theorem 3.2 (Kelvin's Circulation Theorem). *If the flow is isentropic and there is no external force, then the circulation, Γ_{C_t} , is constant.*

Proof.

$$\frac{d}{dt} \Gamma_{C_t} = \frac{d}{dt} \oint_{C_t} \mathbf{u} \cdot ds = \oint_{C_t} \frac{D\mathbf{u}}{Dt} \cdot ds = - \oint_{C_t} \nabla h \cdot ds = 0,$$

where we have used Lemma 3.1, the motion of the isentropic flow $\frac{D}{Dt} \mathbf{u} = -\nabla h$ and the fact that the line integral of a potential vector field along a closed curve is zero. □

Note that this property is due to the lack of viscosity. Since there is no tangential force in the isentropic flow, the rotation of the flow is neither starting nor stopping.

Proposition 3.3. *Let $\omega := \xi/\rho$ be the specific vorticity. For an isentropic flow, we have*

$$\frac{D\omega}{Dt} - (\omega \cdot \nabla) \mathbf{u} = 0 \quad (3.3)$$

and

$$\omega^t(\Gamma(\mathbf{y}, t), t) = D(\Gamma(\mathbf{y}, t)) \omega^t(\mathbf{y}, 0). \quad (3.4)$$

Proof. The derivation of (3.3) is basically the vector calculus. The isentropic flow satisfies

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} \stackrel{?}{=} \mathbf{u}_t + \frac{1}{2} \nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \xi = -\nabla w.$$

Taking the curl gives

$$0 = \xi_t - \text{curl}(\mathbf{u} \times \xi) \stackrel{?}{=} \frac{D\xi}{Dt} - (\xi \cdot \nabla) \mathbf{u} + \xi(\nabla \cdot \mathbf{u}).$$

Since

$$\frac{D\omega}{Dt} = \frac{D}{Dt} \left(\frac{\xi}{\rho} \right) = \frac{1}{\rho} \frac{D\xi}{Dt} + \frac{\xi}{\rho} (\nabla \cdot \mathbf{u}),$$

we obtain (3.3).

Now we show (3.4). Let $F(\mathbf{y}, t) := \omega(\Gamma(\mathbf{y}, t), t)$ and $G(\mathbf{y}, t) := D(\Gamma(\mathbf{y}, t)) \omega^t(\mathbf{y}, 0)$. Then $F^t(\mathbf{y}, 0) = G(\mathbf{y}, 0)$ since $D(\Gamma(\mathbf{y}, 0))$ is the identity matrix. From (3.3),

$$\frac{\partial}{\partial t} F(\mathbf{y}, t) = \frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} = (F \cdot \nabla) \mathbf{u}.$$

On the other hand, by the chain rule:

$$\frac{\partial}{\partial t} G(\mathbf{y}, t) = D_{\mathbf{y}}(\mathbf{u}(\Gamma(\mathbf{y}, t), t)) \omega^t(\mathbf{y}, 0) = D_{\mathbf{x}}(\mathbf{u}(\mathbf{x}, t)) D(\Gamma(\mathbf{y}, t)) \omega^t(\mathbf{y}, 0) = (G \cdot \nabla) \mathbf{u}.$$

Since F, G have the same initial value and satisfy the same first order equation. Hence they are identical. □

For any given domain W_0 , the mass conservation and change of variables give

$$\int_{W_0} \rho(\mathbf{x}, 0) d\mathbf{x} = \int_{W_t} \rho(\mathbf{x}, t) d\mathbf{x} = \int_{W_0} \rho(\mathbf{x}(\mathbf{y}, t), t) |D\Gamma(\mathbf{y}, t)| d\mathbf{y}.$$

Hence, we have

$$\rho(\mathbf{x}, 0) = \rho(\mathbf{x}(\mathbf{y}, t), t) |D\Gamma(\mathbf{y}, t)|,$$

which shows a close relation with (3.4).

3.2.1. *Stokes' Theorem.*

$$\Gamma_C = \int_C \mathbf{u} \cdot ds = \int \int_{\Sigma} (\nabla \times \mathbf{u}) \cdot \mathbf{n} dA = \int \int_{\Sigma} \xi \cdot \mathbf{n} dA, \quad (3.5)$$

where C is an oriented smooth contour of the surface Σ and \mathbf{n} is the unit vector normal to the surface with the same orientation of C . The Circulation Theorem implies that

$$\frac{d}{dt} \int \int_{\Sigma_t} \xi \cdot \mathbf{n} dA = \frac{d}{dt} \Gamma_{C_t} = 0$$

if the flow is isentropic. In other words the flux of the vorticity across a surface moving along the flow is constant in time.

If a surface or a line is tangent to the vorticity vector field ξ , then it is called a **vortex sheet** or a **vortex line** respectively. In particular, if the vortex sheet is like a tube, it is called a **vortex tube**. For example, an integral curve of the vorticity vector field ξ that emanates from a given point \mathbf{x}_0 is a vortex line. Suppose that a curve is given that is nowhere tangent to the vorticity vector field. The collection of integral curves that emanate from the given curve is a vortex sheet. Consider a two dimensional disk-like domain D which is nowhere tangent to ξ . Then the vortex sheet corresponding to the closed curve $C = \partial D$ is called a vortex tube.

Remark 3.4. The above definition of the vortex sheet is sloppy. For a more rigorous definition, we may need to say that the disk D is diffeomorphic to a disc. Furthermore, we need to see that the surface consists of integral curves is diffeomorphic to the tube which is a product of a disc and a real line.

Proposition 3.5. *Let S be a vortex sheet. Then $S_t = \Gamma(S, t)$ is also a vortex sheet for all $t \geq 0$ if the flow is isentropic.*

Proof. Due to the Stokes' theorem (3.5) and Circulation Theorem,

$$\int_{\tilde{S}} \xi \cdot \mathbf{n} dA = 0 \quad \text{for any } \tilde{S} \subset S_t.$$

Hence $\xi \cdot \mathbf{n} = 0$ on S_t . □

Theorem 3.6 (Helmholtz's Theorem). *Assume the flow is isentropic.*

- (1) *If C_1 and C_2 are any two curves encircling a given vortex tube with the same direction, then*

$$\int_{C_1} \mathbf{u} \cdot ds = \int_{C_2} \mathbf{u} \cdot ds.$$

- (2) *This common value is called the strength of the vortex tube and is constant in time, as the tube moves with the fluid.*

Proof. Let S_1 and S_2 be a cross section of the tube such that $C_1 = \partial S_1$ and $C_2 = \partial S_2$ and let S be the part of the tube enclosed by C_1 and C_2 . Let V be the corresponding volume of the tube and $\Sigma = \partial V$ so that $\Sigma = S \cup S_1 \cup S_2$. Using the divergence theorem gives

$$0 = \int_V \nabla \cdot \xi dx = \int_{\Sigma} \xi \cdot dA = \int_{S_1 \cup S_2} \xi \cdot dA.$$

By the Stokes' theorem,

$$\int_{C_1} \mathbf{u} \cdot ds = \int_{S_1} \xi \cdot dA, \quad \int_{C_2} \mathbf{u} \cdot ds = - \int_{S_2} \xi \cdot dA.$$

So (1) holds. (2) follows from the Kelvin's circulation theorem. \square

3.2.2. Two dimensional isentropic flow. When we consider a two dimensional flow, we actually consider a three dimensional flow which is constant with respect to the z - variable.

Exercise 3.7. Show that, for an isentropic flow in the two-dimensional space, $\mathbf{u} = (u_1(x, y, t), u_2(x, y, t), 0)$, $\xi = (0, 0, \xi_3)$ $\Gamma = (\Gamma_1(x, y, t), \Gamma_2(x, y, t), z)$ and

$$\frac{\xi_3}{\rho}(\Gamma(\mathbf{y}, t), t) = \frac{\xi_3}{\rho}(\mathbf{y}, 0), \quad (3.6)$$

$$\frac{D\xi_3}{Dt} = \xi_t + (\mathbf{u} \cdot \nabla)\xi = 0. \quad (3.7)$$

Now consider **incompressible flow** in a simply connected domain D . Then, $u_x + v_y = 0$ and hence the vector field $(u, -v)$ is a potential flow. Let ψ satisfies

$$u = \psi_y, \quad -v = \psi_x.$$

One can easily see that the equi-potential line or the level curve of ψ is a stream line and hence ψ is also called the stream function. Let $(x(s), y(s))$ be a stream line, i.e., $x'(s) = u$ and $y'(s) = v$. Then,

$$\frac{d}{ds} \psi(x(s), y(s), t) = x'(s)\psi_x + y'(s)\psi_y = -uv + vu = 0,$$

which says that the stream line $(x(s), y(s))$ is a level curve of the potential function ψ . Furthermore, one can easily check that

$$\xi_3 = v_x - u_y = -\psi_{xx} - \psi_{yy} = -\Delta\psi.$$

Hence ξ_3 satisfies

$$\frac{D\xi_3}{Dt} = (\xi_3)_t + (\mathbf{u} \cdot \nabla)(\xi_3) = 0, \quad (3.8)$$

$$\Delta\psi = \xi_3, \quad (3.9)$$

$$u = \psi_y, \quad -v = \psi_x, \quad (3.10)$$

$$\psi = 0 \text{ on } \partial D. \quad (3.11)$$

Consider

$$(\mathbf{u} \cdot \nabla)\xi_3 = u(\xi_3)_x + v(\xi_3)_y = \psi_y(\xi_3)_x - \psi_x(\xi_3)_y = \left| \begin{array}{c} \nabla\psi \\ \nabla\xi_3 \end{array} \right|. \quad (3.12)$$

Hence $(\xi_3)_t = 0$, i.e, the flow is stationary if and only if $\nabla\psi$ and $\nabla\xi_3$ are parallel.

Example 3.8 (A Flow with Radial Symmetry). Suppose that the flow is radial symmetric, i.e., $u = u(r)$ and $v = v(r)$ with $r = \sqrt{x^2 + y^2}$. Then, $\nabla\psi$ is a function of r and so is ψ . Furthermore,

$$u = \psi_y = \psi_r r_y = \frac{y}{r} \psi_r, \quad v = -\psi_x = -\psi_r r_x = -\frac{x}{r} \psi_r.$$

Hence the velocity vector is tangent to the circle of radius r and rotating clockwise if $\psi_r > 0$. The scalar vorticity ξ_3 satisfies

$$\xi_3 = -\Delta\psi = -\frac{1}{r}(r\psi_r)_r,$$

which is a function of r only. Suppose that $\psi_r \neq 0$ for all $r > 0$. Then one can easily check that the determinant in (3.12) is zero and hence the flow is stationary. In other words an incompressible isentropic flow in two dimensional space is stationary if it is radial symmetric. (Q: Do we need $\psi \neq 0$ for this conclusion?)

4. SIMILARITY & SCALING INVARIANCE

The logic we are going to employ to survey the similarity structure of several kinds of PDEs is the following. The preliminary requirement is that the problem under consideration should be well-posed, or the uniqueness should be guaranteed at least. Let $u(x, t)$ be a solution to such a problem and consider a rescaled function v given by

$$u(x, t) = av(bx, ct), \quad a, b, c > 0. \quad (4.1)$$

The main step is to find the relation among a, b, c that guarantees for v to be the solution of the same problem. Then the uniqueness of the problem implies $u = v$ and hence

$$u(x, t) = au(bx, ct), \quad (4.2)$$

which indicates that the solution should have certain structure provided by the equation. Survey of such behavior is the purpose of this section.

4.1. Riemann Problem. The first example is the Riemann problems of conservation laws given by

$$\begin{aligned} u_t + f(u)_x &= 0, \\ \lim_{t \downarrow 0} u(x, t) &= \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \end{aligned} \quad (4.3)$$

One can easily see that the rescaled function v satisfies the initial value

$$\lim_{t \downarrow 0} v(x, t) = \begin{cases} au_l, & x < 0, \\ au_r, & x > 0. \end{cases}$$

Therefore, if v is expected to be the solution of the Riemann problem (4.3), then one should pick $a = 1$. Now let's find the relation between b and c . One can easily check that

$$u_t = cv_t, \quad f'(u)u_x = bf'(v)v_x \quad \text{and} \quad u_t + f(u)_x = cv_t + bf(v)_x = 0.$$

Therefore, if $b = c$, then v satisfies the conservation law $v_t + f(v)_x = 0$. The solution of the conservation law that satisfies the entropy condition is unique. For example for the convex scalar case the entropy condition implies that the solution satisfy

$$\lim_{x \uparrow x_0} u(x, t) \geq \lim_{x \downarrow x_0} u(x, t).$$

Since $b > 0$, such relations are not changed and v also satisfies the entropy condition. Since the solution is unique we may conclude $v(x, t) = u(x, t)$ and, hence,

$$u(x, t) = v(bx, bt) = u(bx, bt). \quad (4.4)$$

This relation indicates that the solution u is constant along any line that emanates from the origin. (See Figure 1.) In other words u is a function of x/t . Therefore, it is natural to introduce a new variable

$$\xi = \frac{x}{t} \quad (4.5)$$

One may say that the problem (4.3) is invariant under the scaling $u(x, t) \rightarrow u(bx, bt)$ and the variable ξ also has such a scaling invariance, i.e., even if we substitute bx in the place of x and bt in the place of t , ξ is not changed:

$$\frac{bx}{bt} = \frac{x}{t} = \xi.$$

Therefore, one call this variable ξ a similarity variable of the Riemann problem (4.3). One can easily rewrite the equation in terms of this similarity variable. Consider

$$\xi_x = t^{-1}, \quad \xi_t = -\xi t^{-1}, \quad \frac{\partial}{\partial t} u(\xi) = u' \xi_t = -u' \xi t^{-1}, \quad \frac{\partial}{\partial x} f(u) = f(u)_{\xi} \xi_x = f'(u) u' t^{-1}.$$

Therefore, the problem (4.3) is transformed to a one dimensional boundary problem:

$$\begin{aligned} -\xi u' + f'(u) u' &= 0, \\ \lim_{t \rightarrow -\infty} u(\xi) &= u_l, \quad \lim_{\xi \rightarrow +\infty} u(\xi) = u_r. \end{aligned} \quad (4.6)$$

Notice that we use almost nothing on the flux function $f(u)$. It can be a system of any kind as long as it provide proper uniqueness property. Therefore, the similarity structure is for systems of conservation laws with Riemann initial data.

HOME WORK. Discuss the similarity structure in the following problem.

$$\begin{aligned} u_t &= u_{xx}, \\ \lim_{t \downarrow 0} u(x, t) &= \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases} \end{aligned}$$

4.2. L^1 solutions to convection equations. Let $u(x, t)$ be an L^1 solution to

$$\begin{aligned} u_t + f(u)_x &= 0, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned}$$

Since the total mass is preserved, one should consider a scaling that preserve the total mass. If not, there is no chance that v and u are identical. Therefore, we set $a = b$, i.e., the scaling considered is

$$u(x, t) = av(ax, ct), \quad a, c > 0. \quad (4.7)$$

Unlike the Riemann problem case the relation of scaling of invariance depends on the flux function f since $a \neq 1$. Therefore, we can not consider the flux function with the generality of the Reimann problem. Here, we consider a scalar case under a power law:

$$\begin{aligned} u_t + f(u)_x &= 0, \quad f(u) = |u|^q/q, \quad q > 1 \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \quad (4.8)$$

One can easily check that

$$u_t = acv_t, \quad u_x = a^2v_x \quad \text{and} \quad u_t + f'(u)u_x = acv_t + a^{q+1}f'(v)v_x = 0.$$

Therefore, if $c = a^q$, then v also satisfies the same convection equation. Now we consider similarity variables. Consider

$$\xi = xt^{-1/q}, \quad u(x, t) = t^{-1/q}\tilde{u}(\xi, t).$$

Then the quantities ξ and \tilde{u} are not changed under the rescaling $(x, t) \rightarrow (ax, a^qt)$ and $u \rightarrow u/a$. One may derive an equation satisfied by these similarity variables. Using the chain rule, one may obtain

$$\xi_x = t^{-1/q}, \quad \xi_t = -\xi t^{-1/q}, \quad u_x = t^{-2/q}\tilde{u}_\xi, \quad u_t = t^{-1/q}t^{-1}\tilde{u}/q - t^{-1/q}\tilde{u}_\xi\xi t^{-1/q} + t^{-1/q}\tilde{u}_t.$$

Substituting these quantities to the equation we obtain

$$u_t + \text{sign}(u)|u|^{q-1}u_x = t^{-1/q}(\tilde{u}_t + t^{-1}(\text{sign}(\tilde{u})|\tilde{u}|^{q-1}\tilde{u}_\xi - (\xi\tilde{u}/q)_\xi)) = 0.$$

Therefore, \tilde{u} satisfies

$$\tilde{u}_t + t^{-1}(|\tilde{u}|^q/q - \xi\tilde{u}/q)_\xi = 0.$$

Since the dependent factor t^{-1} appears explicitly in the equation, people consider a new time scale $s = \ln t$. The full similarity variables are

$$\xi = \frac{x}{\sqrt[q]{t}}, \quad s = \ln t, \quad \tilde{u}(\xi, s) = \sqrt[q]{t}u(x, t). \quad (4.9)$$

Under the is exponentially fast time scale, $t = e^s$, the only difference is $\tilde{u}_t = \tilde{u}_s t^{-1}$ and the equation is turned into

$$\tilde{u}_s + (|\tilde{u}|^q/q - \xi\tilde{u}/q)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1). \quad (4.10)$$

Notice that $s = 0$ corresponds to $t = 1$ and, hence we have $\tilde{u}(\xi, 0) = u(\xi, 1)$. It is also common use the variables $s = \ln(t + 1)$, $\xi = x/\sqrt[q]{t + 1}$, $\tilde{u} = \sqrt[q]{t + 1}u$. In the case one have $\tilde{u}(\xi, 0) = u(\xi, 0)$ and the computation is slightly more complicate.

We have checked that the function $v(x, t)$ given by

$$u(x, t) = av(ax, a^qt)$$

satisfied the differential equation (4.8). Therefore, one may ask if $u = v$. To see that one should check the initial value. Under the above scaling one can easily check that

$$v(x, 0) = u(x/a, 0)/a.$$

Therefore, the only L^1 function that provides the identical initial value is the dirac-measure-type $M\delta(x)$. Such a solution is usually called a source-type solution and we denote this source-type solution by $\rho(x, t)$, i.e., $\rho(x, t)$ is a solution having initial value

$$\lim_{t \downarrow 0} \rho(x, t) = M\delta(x). \quad (4.11)$$

Similarly set

$$\tilde{\rho}(\xi, s) = \sqrt[q]{t}\rho(x, t).$$

If positive solutions are considered, then the source-type solution is unique. However, there is no unique solution if sign-changing solutions are allowed. In that case two invariant variables are introduced, which are

$$P = -\inf_x U(x, t), \quad Q = P + \lim_{x \rightarrow \infty} U(x, t), \quad \text{where} \quad U(x, t) = \int_{-\infty}^x u(y, t)dy. \quad (4.12)$$

It is well known that the positive quantities P and Q are constant in time if the flux is convex, $f''(u) \geq 0$, and $Q - P = M$. If these invariant constants are prescribed, then the source-type solution is unique and hence we may conclude that,

$$\text{if } P \text{ and } Q \text{ are prescribed, then } \rho(x, t) = a\rho(ax, a^q t).$$

Therefore, the value of $\rho(x, t)$ along the curve $x/\sqrt[q]{t} = \text{constant}$ is decided completely by the value at a single point, and the rescaled value $\tilde{\rho}$ is constant along it. (See Figure 2.) Therefore, $\tilde{\rho}$ is a function of ξ only if the initial value is given by the dirac-measure. This similarity solution is computed explicitly in the followings. Since $\tilde{\rho}_s = 0$ and $\tilde{\rho}$ is compactly supported,

$$(|\tilde{\rho}|^q - \xi\tilde{\rho})_\xi = 0 \Rightarrow |\tilde{\rho}|^q - \xi\tilde{\rho} = 0 \Rightarrow \tilde{\rho} = 0 \text{ or } \text{sign}(\tilde{\rho})|\tilde{\rho}|^{q-1} = \xi. \quad (4.13)$$

The solution that has the invariants P, Q and that satisfies the entropy condition should be given by

$$\tilde{\rho}(\xi) = \begin{cases} (f')^{-1}(\xi), & -a_P < \xi < a_Q, \\ 0, & \text{otherwise,} \end{cases} \quad (4.14)$$

where the end points of the support is given by the relation

$$-\int_{-a_P}^0 \tilde{\rho}(\xi)d\xi = P, \quad \int_0^{a_Q} \tilde{\rho}(\xi)d\xi = Q.$$

From these relations, a_P and a_Q are explicitly computed:

$$a_P = \left(\frac{qP}{q-1}\right)^{\frac{q-1}{q}}, \quad a_Q = \left(\frac{qQ}{q-1}\right)^{\frac{q-1}{q}}. \quad (4.15)$$

This self-similar solution is called an N -wave due to its shape. (See Figure 3.) If the positive solutions are considered, then $p = 0$ and $q = M$ and the N -wave has simply a triangle like shape.

Under the change of variable, one can easily transform the N -wave in terms of the original variables:

$$\rho(x, t) = \begin{cases} (f')^{-1}(x/t), & -a_P\sqrt[q]{t} < \xi < a_Q\sqrt[q]{t}, \\ 0, & \text{otherwise,} \end{cases} \quad (4.16)$$

One may consider a flux such as $f(u) = |u|^2 + |u|^3$. In this case the solution has no exact scaling invariance. However, $\rho(x, t)$ in (4.16) is well defined for all convex flux, $f''(u) \geq 0$, and is the source-type solution.

Since $\max_x \tilde{\rho}$ is of order $O(1)$, the source-type solution in the original variable has the order

$$\max_x \rho(x, t) = O(t^{-1/q}) \quad \text{as } t \rightarrow \infty. \quad (4.17)$$

4.3. L^1 solutions to diffusion equations. Let $u(x, t)$ be a positive L^1 solution to a nonlinear diffusion equation

$$\begin{aligned} u_t &= \phi(u)_{xx}, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}), \end{aligned}$$

where the diffusion function ϕ is given by

$$\phi(u) = u^m, \quad m > 0. \quad (4.18)$$

Consider a scaling of

$$u(x, t) = av(ax, ct), \quad a, c > 0. \quad (4.19)$$

One can easily check that

$$u_t = acv_t, \quad \phi(u)_{xx} = a^{m+2}\phi(v)_{xx} \quad \text{and} \quad u_t - \phi(u)_{xx} = acv_t - a^{m+2}\phi(v)_{xx} = 0. \quad (4.20)$$

Therefore, if $c = a^{m+1}$, then v also satisfies the same diffusion equation. Now we consider similarity variables. Consider

$$\alpha = m + 1, \quad s = \ln t, \quad \xi = xt^{-1/\alpha}, \quad u(x, t) = t^{-1/\alpha}\tilde{u}(\xi, s).$$

Then the quantities ξ and \tilde{u} are not changed under the rescaling $(x, t) \rightarrow (ax, a^\alpha t)$ and $u \rightarrow u/a$. One may derive an equation satisfied by these similarity variables. Using the chain rule, one obtain

$$\xi_x = t^{-\frac{1}{\alpha}}, \quad \xi_t = -\frac{\xi t^{-1}}{\alpha}, \quad \phi(u)_{xx} = t^{-\frac{2}{\alpha}}t^{-1}\phi(\tilde{u})_{\xi\xi}, \quad u_t = -\frac{t^{-\frac{1}{\alpha}}t^{-1}}{\alpha}\tilde{u} - \frac{t^{-\frac{1}{\alpha}}t^{-1}}{\alpha}\xi\tilde{u}_\xi + t^{-\frac{1}{\alpha}}t^{-1}\tilde{u}_s.$$

Substituting these quantities to the equation we obtain

$$u_t - \phi(u)_{xx} = t^{-1-1/\alpha}(\tilde{u}_s - (\phi(u)_\xi + \xi\tilde{u}/\alpha)_\xi) = 0.$$

Therefore, $\tilde{u}(\xi, s)$ satisfies

$$\tilde{u}_s - (\phi(u)_\xi + \xi\tilde{u}/\alpha)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1). \quad (4.21)$$

Let $\rho(x, t)$ be the positive source-type solution with initial value $\lim_{t \downarrow 0} u(x, t) = M\delta(x)$. Then, due to the similarity property its similarity quantity $\tilde{\rho}(\xi, s)$ is a function of ξ only and satisfies

$$(\phi(\tilde{\rho})_\xi + \xi\tilde{\rho}/\alpha)_\xi = 0.$$

We may do the similar computation as the convection case:

$$(\phi(\tilde{\rho})_\xi + \xi\tilde{\rho}/\alpha)_\xi = 0 \Rightarrow \phi(\tilde{\rho})_\xi + \xi\tilde{\rho}/\alpha = 0 \Rightarrow \rho = 0 \quad \text{or} \quad \frac{\phi'(\tilde{\rho})}{\tilde{\rho}}\tilde{\rho}_\xi + \frac{\xi}{\alpha} = 0 \Rightarrow \tilde{\rho} = 0 \quad \text{or} \quad g(\tilde{\rho})_\xi = -\frac{\xi}{\alpha},$$

where

$$g'(x) = \phi'(x)/x, \quad (\text{i.e., } g(x) = \frac{m}{m-1}x^{m-1} \text{ for } \phi(x) = x^m).$$

Therefore, we may conclude that

$$\tilde{\rho}(\xi) = \max(0, g^{-1}(C - \xi^2/(2\alpha))), \quad (4.22)$$

where C is a constant that should be decided by the total mass relation $\int \tilde{\rho}(\xi)d\xi = M$.

Under the change of variable, one can easily transform the N-wave in terms of the original variables:

$$\rho(x, t) = \max(0, g^{-1}(Ct^{\frac{1-m}{m+1}} - x^2/(2\alpha t))) = \max(0, g^{-1}([Ct^{\frac{2}{m+1}} - x^2/(2\alpha)]/t)). \quad (4.23)$$

Notice that, unlike the convection case, the expression of $\rho(x, t)$ depends on the power m . Therefore, it is not obvious to find the explicit formula for the source-type solution for a general case when the diffusion is not given by the power law. we are not sure if we may find one.

Since $\max_x \tilde{\rho}$ is of order $O(1)$, the source-type solution in the original variable has the order

$$\max_x \rho(x, t) = O(t^{-1/(m+1)}) \quad \text{as } t \rightarrow \infty. \quad (4.24)$$

HOME WORK.

- (1) Show that the self-similar solution $\rho(x, t)$ is the Gaussian for the linear heat equation case $m = 1$.
- (2) Show that the support of $\rho(x, t)$ is compact for any given time $t > 0$ if $m > 1$.
- (3) Show that the support of $\rho(x, t)$ is strictly positive compact on the whole real line for any given time $t > 0$ if $m > 1$.
- (4) Discuss the similarity structure for the multidimensional case

$$u_t = \Delta u^m, \quad m > 0.$$

4.4. **p-Laplacian.** Let $u(x, t)$ be a positive L^1 solution to a different kind of non-linear diffusion equation, so called the p-Laplacian,

$$\begin{aligned} u_t &= (|u_x|^{p-2} u_x)_x, \quad p > 1, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \quad (4.25)$$

One may similarly study the similarity structure of the p-Laplacian equation. From the similarity variables one can show the asymptotic magnitude of the source-type solution:

$$\max_x \rho(x, t) = O(t^{-1/(2p-2)}) \quad \text{as } t \rightarrow \infty. \quad (4.26)$$

HOME WORK.

- (1) Find the similarity variables of the solutions to (4.25).
- (2) Find the equation that the similarity variables satisfy.
- (3) Find the explicit formula that the self-similar solution satisfies.
- (4) Verify that the self-similar solution $\rho(x, t)$ is the Gaussian for the linear heat equation case $p = 2$.
- (5) Find the range of the power $p > 1$ that the self-similar solution $\rho(x, t)$ is compactly supported.
- (6) Discuss the similarity structure for the multidimensional case

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad p > 1.$$

One can easily check that

$$u(x, t) = av(ax, a^{2p-2}t)$$

is the invariance scaling of the p-Laplacian. Therefore, the similarity variables are

$$\alpha = 2p - 2, \quad \xi = xt^{1/\alpha}, \quad s = \ln t, \quad \tilde{u}(\xi, s) = t^{1/\alpha} u(x, t) \quad (4.27)$$

and they satisfy

$$\tilde{u}_s - (|\tilde{u}_\xi|^{p-2} \tilde{u}_\xi + \xi \tilde{u}/\alpha)_\xi = 0, \quad \tilde{u}(\xi, 0) = u(\xi, 1).$$

Let $\rho(x, t)$ be the source-type solution and $\tilde{\rho}$ be the self-similar correspondence. Then, $\tilde{\rho}$ is the solution of the ordinary differential equation

$$|\tilde{\rho}_\xi|^{p-2} \tilde{\rho}_\xi + \xi \tilde{\rho}/\alpha = 0.$$

How to solve it?

4.5. Co-existence of several different phenomena. Let $u(x, t)$ be a positive L^1 solution to a different kind of the following mixed problem,

$$\begin{aligned} u_t &= (u^m)_{xx} + (|u_x|^{p-2}u_x)_x - (u^q/q)_x, \quad m > 0, q > 1, p > 1, \\ \lim_{t \downarrow 0} u(x, t) &= u_0(x) \in L^1(\mathbf{R}). \end{aligned} \quad (4.28)$$

If

$$q = m + 1 = 2p - 2,$$

then the problem has a invariance scaling and hence we may consider similarity structure of the problem. If not, there is no exact similarity property. However, one may find the dominant factor of the evolution. We may conclude that

$$\max_x |u(x, t)| = O(t^{-1/\alpha}) \quad \text{as } t \rightarrow \infty, \quad \alpha = \min(q, m + 1, 2p - 2). \quad (4.29)$$

If $\alpha = q$, then we may say that the convection dominates the evolution. Similarly the PME type diffusion dominates the evolution if $\alpha = m + 1$ and the p-Laplacian type diffusion decide the asymptotics if $\alpha = 2p - 2$.

HOME WORK.

- (1) Suppose that $\alpha = q$ in (4.29). Transform the equation (4.28) using the similarity variables for the convection case (4.9) and observe that the coefficients for other terms decays to zero.
- (2) Discuss what decide the similarity structure of the problems. What is the basic difference in the influence of convection and diffusion when they decide the scaling invariance.
- (3) When the magnitude of the solution decays faster than the order $1/t$?

5. CHARACTERISTICS FOR CONSERVATION LAWS

In this section we consider convection equations. Let $u(x, t)$ be the entropy solution to the convection equation

$$u_t + a(x, t)u_x = 0, \quad \lim_{t \downarrow 0} u(x, t) = u_0(x) \in L^1(\mathbf{R}). \quad (5.1)$$

The characteristics is a curve $x = x(t)$ in the $x - t$ plane that satisfies

$$x'(t) = a(x(t), t).$$

Then one can easily check that

$$\frac{d}{dt}u(x(t), t) = u_t + x'(t)u_x = u_t + a(x(t), t)u_x = 0.$$

Therefore, u is constant along the characteristic curve $x = x(t)$,

5.1. scalar conservation laws. Consider

$$u_t + f(u)_x = u_t + f'(u)u_x = 0 \quad (5.2)$$

under the assumption on the smooth flux of

$$f(0) = f'(0) = 0, \quad f''(u) \geq 0.$$

In this case $x'(t) = f'(u)$ and u is constant along the characteristics. Therefore, the slope of the curve is constant, i.e, characteristics are strait lines. (If the waves speed $a(x, t)$ is discontinuous, then we consider minimal or maximal characteristics.)

If characteristics collide to each other, then discontinuities may appear. In the case the speed of the propagation of the discontinuity, or shock speed, is given by the Rankine-Hugoniot jump condition,

$$\text{shock speed } s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad \text{and } s = \frac{f(u)}{u} \text{ if } u_- = 0.$$

We employ the primitive of the solution as its potential:

$$U(x, t) = \int_{-\infty}^x u(y, t) dy. \quad (5.3)$$

Integrating (5.3) on $(-\infty, x)$, one obtains

$$U_t + f(u) = U_t + \frac{f(u)}{u} U_x = 0$$

Here, potential U is constant along the characteristic $x'(t) = f(u)/u$. Notice that this curve is not the same the characteristic of u and is not a straight line any more. This curve is related to the shock characteristics when one state is connected to the zero state.

5.2. Isentropic Gas Dynamics. In this section $\rho(x, t)$ represents the density of the given gas, $u(x, t)$ is the velocity, m is the momentum, $m = \rho u$, and p is the pressure. In the isentropic gas the pressure is assumed to be a function of density only $p = p(\rho)$ and they satisfies the equation

$$\begin{aligned} \rho_t + m_x &= 0, \\ m_t + (m^2/\rho + p(\rho))_x &= 0. \end{aligned} \quad (5.4)$$

Set potentials as

$$R(x, t) = \int_{-\infty}^x \rho(y, t) dy, \quad M(x, t) = \int_{-\infty}^x m(y, t) dy.$$

Then after the integration of the equation (5.4), one obtain

$$\begin{aligned} R_t + uR_x &= 0, \\ M_t + (u + p(\rho)/m)M_x &= 0. \end{aligned} \quad (5.5)$$

The first relation in (5.5) implies that R is constant along the curve

$$x'_1(t) = u|_{(x_1(t), t)}.$$

That is natural since the characteristic of the first class is the trace of a moving particle and the total mass of the gas on the left hand side of it should be constant.

Under the power law $p(\rho) = \rho^\gamma$, $\gamma \geq 1$, the characteristic curve for the momentum potential is

$$x'_2(t) = (u + \rho^{\gamma-1}/u)|_{(x_2(t), t)}.$$

This indicates that the characteristic curve on which the momentum potential is constant has higher speed than the particle trace has. In particular, if the velocity u is near zero, then the characteristic speed increases to infinity and, if the density ρ approaches to zero, two curves $x_{1,2}(t)$ approaches to each other.

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 335 GWAHANGNO, YUSEONG-GU, DAEJEON 305-701, REPUBLIC OF KOREA

E-mail address: yongkim@kaist.edu