FLOCKING DYNAMICS OF SINGULARLY PERTURBED OSCILLATOR CHAIN AND THE CUCKER-SMALE SYSTEM

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Dedicated to Prof. J.K. Hale on the occasion of his eightieth birthday.

ABSTRACT. In this note, we show how the one-dimensional Cucker-Smale system for "*flocking*" dynamics can be recovered as a singular perturbation limit of a chain of damped oscillators.

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1. INTRODUCTION

In the literature of flocking, at least two classes models have appeared. One class (of Cucker-Smale type [3, 4]) is a particle model for a weakly communicating group of self-propelled agents:

$$\begin{aligned} &\frac{dx_i}{dt} = v_i, \quad t \geq 0, \\ &\frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \omega(x_j - x_i)\big(v_j - v_i\big), \quad 1 \leq i \leq N, \end{aligned} \tag{1.1}$$

where x_i and v_i denote the position and velocity of the i-th particle respectively, and λ is a nonnegative coupling constant. Then nonnegative function $\omega(x_j - x_i)$ is a communication rate between i and j-particles and assumed to be symmetric with respect to x_i and x_j.

On the other hand, independently Erdmann, Ebeling and Mikhailov [5] have introduced a particle model based on a chain of damped oscillators and given a description of "swarms" which appeared to be analogous to "flocking behavior".

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In this paper, we attempt to relate these two modeling approaches by showing that the one-dimensional Cucker-Smale system (1.1) can be obtained as a rigorous singular limit of a damped chain of oscillators. The main tool in our analysis is a classic theorem of A. N. Tikhonov [12] on singular perturbations.

This paper is divided into three sections after this introduction. Section 2 defines our damped particle chain on a line and note that its formal singular limit is the Cucker-Smale system. Section 3 provides a rigorous proof of a singular limit via the classical theorem of Tikhonov, and finally Section 4 notes that of course flocking estimate for the Cucker-Smale system must imply a flocking estimate for the particle chain model as well.

2. A mechanical model of particles on a line

In this section, we discuss a damped particle chain and its formal singular limit to the Cucker-Smale flocking system.

Motivated by the work of Erdmann, Ebeling and Mikhailov[5], we consider a simple mechanical model of particles on a line with a constant friction coefficient ($\varepsilon > 0$):

$$\frac{dx_i}{dt} = v_i, \tag{2.1a}$$

$$\frac{dv_i}{dt} = \frac{\lambda}{\varepsilon N}\sum_{j=1}^{N} p'(x_j - x_i) - \frac{1}{\varepsilon} v_i. \tag{2.1b}$$

Here p is the pairwise interaction potential, and is assumed to be a twice continuously differential convex function. The function $\omega = p''$ is called the communication rate.

If we differentiate the equation (2.1b), we obtain

$$\frac{d^2 v_i}{dt^2} = \frac{\lambda}{\varepsilon N}\sum_{j=1}^{N} p''(x_j - x_i)(v_j - v_i) - \frac{1}{\varepsilon}\frac{dv_i}{dt}. \tag{2.2}$$

Its formal limit as $\varepsilon \to 0$ yields

$$\frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \omega(x_j - x_i)(v_j - v_i), \qquad \omega(x_j - x_i) := p''(x_j - x_i). \tag{2.3}$$

The equation (2.3) when coupled with the equation (2.1a) is the standard Cucker-Smale system on a line [3, 4, 6, 9]. Our goal is now to give a rigorous proof of this limiting process, thereby establishing a connection between the mechanical particle system (2.1a)-(2.1b) and the Cucker-Smale system (1.1). For any p satisfying

$$p'(s) = \int^{s} \omega(|\tau|)d\tau, \quad \omega \geq 0,$$

may be used in (2.3). The nonnegativity and symmetric property of the communication function ω is crucial in the flocking analysis. This can be easily seen in the following example.

Example. Consider the two particle system on a line with anti-symmetric communication rate $\omega(x_j - x_i) = \frac{1}{x_j - x_i}$:

$$\frac{dx_1}{dt} = v_1, \quad \frac{dx_2}{dt} = v_2,$$

$$\frac{dv_1}{dt} = \frac{\lambda(v_2 - v_1)}{2(x_2 - x_1)}, \quad \frac{dv_2}{dt} = \frac{\lambda(v_1 - v_2)}{2(x_1 - x_2)}.$$

Then it is easy to check that the differences for position and velocity satisfy

$$\frac{d}{dt}(x_2 - x_1) = v_2 - v_1, \qquad \frac{d}{dt}(v_2 - v_1) = 0.$$

These yield

$$(v_2 - v_1)(t) = (v_2 - v_1)(0), \qquad (x_2 - x_1)(t) = (x_2 - x_1)(0) + (v_2 - v_1)(0)t,$$

which means non-flocking of two-particle system.

On the other hand, the Erdmann, Ebeling and Mikhailov model is simply (2.1a) and (2.1b) with $p'(s) = s$ and the linear damping replaced by the sum of a nonlinear damping $(1 - v_i^2)v_i$ and a stochastic white forces $\xi_i(t)$ which are independent and are characterized by the mean zero and correlation relation:

$$\langle \xi_i \rangle = 0, \qquad \langle \xi_i(t)\xi_j(t') \rangle = 2D\delta(t - t')\delta_{ij},$$

where $\langle \cdot \rangle$ is a ensemble average.

Thus we see that Erdmann, Ebeling and Mikhailov accentuate the role played by the nonlinear damping and stochastic noises, whereas here we emphasize the role by a general nonlinear communication role.

3. The particle system as a fast-slow dynamical system

In this section, we show that the Cucker-Smale system (1.1) can be derived from the damped mechanical system (2.1a) and (2.2) as a direct application of Tikhonov's theorem.

We first rewrite the particle system (2.1a) and (2.2) as a first order system:

$$\frac{dx_i}{dt} = v_i, \quad 1 \le i \le N, \tag{3.1a}$$

$$\frac{dv_i}{dt} = z_i, \tag{3.1b}$$

$$\varepsilon\frac{dz_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \omega(x_j - x_i)(v_j - v_i) - z_i. \tag{3.1c}$$

In the language of a singular perturbation theory, x_i and v_i are slow variables, and z_i are fast variables. For reader's convenience, we recall the classical theorem on the singular perturbation limit due to A. N. Tikhonov.

Consider the slow-fast dynamical system:

$$\begin{aligned} \frac{dy_i}{dt} &= f_i(y, z, t), \quad i = 1, , n, \\ \mu_j\frac{dz_j}{dt} &= F_j(y, z, t), \quad j = 1, \cdots, m, \end{aligned} \tag{3.2}$$

where $y = (y_1, \cdots, y_n), z = (z_1, \cdots, z_m)$, and μ_j are small positive parameters depending on a parameter μ in such a way that

$$\lim_{\mu\to 0} \mu_j(\mu) = 0, \quad \lim_{\mu\to 0} \frac{\mu_{j+1}}{\mu_j} = 0 \quad \text{or} \quad 1.$$

Theorem 3.1. Tikhonov [12] *Suppose the following conditions hold.*

(1) *The degenerate system obtained by setting all* $\mu_j = 0$

$$\frac{dy_i}{dt} = f_i(y,z,t), \qquad F_j(y,z,t) = 0$$

have continuous solutions.

(2) *The roots* $z_j = \psi_j(y,t)$ *of* $F_j(y,z,t) = 0$ *have continuous first partial derivatives and are stable, i.e. the expression* $\sum_j (z_j - \psi_j(y,t))F_j(y,z,t)$ *is negative in a suitable deleted neighborhood* $\mathcal{N}$ *of the roots.*

Then as $\mu \to 0$*, the solution of* (3.2) *tend to the corresponding solutions of the degenerate system with the initial data* $(y_i^0, z_j^0, t^0) \in \mathcal{N}$*, and this convergence is uniform for* $t \geq t_1 > t^0$*.*

Then direct application of Tikhonov's theorem, we obtain the following main result of this paper.

Theorem 3.2. *Assume that the initial conditions* $(x_i^\varepsilon(0), v_i^\varepsilon(0), z_i^\varepsilon(0))$ *of the system* (3.1a) - (3.1c) *satisfy*

$$\lim_{\varepsilon \to 0} \Big\{ z_i^\varepsilon(0) - \frac{\lambda}{N} \sum_{j=1}^N \omega(x_j^\varepsilon(0) - x_i^\varepsilon(0))(v_j^\varepsilon(0) - v_i^\varepsilon(0)) \Big\} = 0. \tag{3.3}$$

Then on the interval $[t_0, \infty), t_0 > 0$*, solutions* $(x_i^\varepsilon(t), v_i^\varepsilon(t), z_i^\varepsilon(t))$ *converge uniformly in* t *to* $(x_i^0(t), v_i^0(t), z_i^0(t))$*, where* $(x_i^0(t), v_i^0(t), z_i^0(t))$ *satisfy the Cucker-Smale system*

$$\frac{dx_i^0}{dt} = v_i^0, \quad t \geq 0, \quad 1 \leq i \leq N, \tag{3.4a}$$

$$\frac{dv_i^0}{dt} = z_i^0, \tag{3.4b}$$

$$z_i^0 = \frac{\lambda}{N} \sum_{j=1}^N \omega(x_j^0 - x_i^0)(v_j^0 - v_i^0). \tag{3.4c}$$

Proof. Notice that our hypothesis on the initial data (3.3) insures that the data for (3.1a), (3.1b) and (3.1c) lie in any open neighborhood $\mathcal{N}$ of the data $(x^0(0), v^0(0), z^0(0))$ for the Cucker-Smale system and hence satisfy Tikhonov's hypothesis on the data. Then the conclusion follows from the direct application of Theorem 3.1. A more modern approach which generalizes the Tikhonov's original idea as well as the method of averaging is in Artstein and Vigodner [2]. □

4. Flocking for Cucker-Smale and singularly perturbed systems

In this section, we review the flocking estimates for the Cucker-Smale system due to Ha and Liu [9] and study the corresponding estimate for the singularly perturbed system. We first recall the definition of flocking for many-body particle system.

Definition 4.1. *The interacting particle system* $\{(x_i, v_i)\}_{i=1}^N$ *has a time-asymptotic flocking if and only if the following two conditions hold.*

(1) *The velocity difference for each pair goes to zero as* $t \to \infty$*:*

$$\lim_{t \to +\infty} |v_i(t) - v_j(t)| = 0.$$

(2) *The diameter of a group is uniformly bounded in t:*

$$\sup_{0\le t<\infty} \sum_{i,j} |x_i(t) - x_j(t)| < \infty.$$

The existence for flocking for the Cucker-Smale system (1.1) has been given under the various conditions on the communication rate ω [3, 4, 6, 9, 11]. Here we recall the most general result known to us. For given $x = (x_1, \cdots, x_N), v = (v_1, \cdots, v_N)$, we set

$$x_{cm} := \frac{1}{N}\sum_{i=1}^N x_i, \qquad v_{cm} := \frac{1}{N}\sum_{i=1}^N v_i,$$
$$||x|| := \Big(\sum_{i=1}^N |x_i|^2\Big)^{\frac{1}{2}}, \qquad ||v|| := \Big(\sum_{i=1}^N |v_i|^2\Big)^{\frac{1}{2}}.$$

Theorem 4.1. [9] *Suppose the communication rate $\omega(x_j - x_i)$ satisfy*

$$\omega(x_j - x_i) = \tilde{\omega}(|x_j - x_i|) \ge 0,$$

where $\tilde{\omega}$ is a non-increasing bounded function. Let (x_i, v_i) be the solution to the system (1.1) *with initial data $(x_i(0), v_i(0))$. Suppose the initial data (x_{0i}, v_{0i}) satisfies*

$$||v_0 - v_{cm}|| < \int_{||x_0 - x_{cm}||}^{\infty} \tilde{\omega}(s)ds.$$

Then we have a flocking in the sense of Definition 4.1.

Remark 4.1. *For the stochastically perturbed Cucker-Smale system and its flocking estimate, we refer to* [8].

As a corollary of Theorem 3.1 and Theorem 4.1, we obtain approximate flocking for the singularly perturbed system (2.1a) - (2.1b).

Corollary 4.1. *Given any $\varepsilon > 0$ sufficiently small, solutions to* (2.1a) *and* (2.1b) *satisfy*

$$\text{(i)} \ \lim_{t\to\infty} |v_i^\varepsilon(t) - v_j^\varepsilon(t)| < \varepsilon.$$
$$\text{(ii)} \ \sup_{0\le t<\infty} |x_i^\varepsilon(t) - x_j^\varepsilon(t)| < \infty.$$

Proof. We simply apply the triangle inequalities

$$|v_i^\varepsilon(t) - v_j^\varepsilon(t)| \le |v_i^\varepsilon(t) - v_i^0(t)| + |v_j^\varepsilon(t) - v_j^0(t)| + |v_i^0(t) - v_j^0(t)|,$$
$$|x_i^\varepsilon(t) - x_j^\varepsilon(t)| \le |x_i^\varepsilon(t) - x_i^0(t)| + |x_j^\varepsilon(t) - x_j^0(t)| + |x_i^0(t) - x_j^0(t)|.$$

Then Theorem 3.1 and Theorem 4.1 yield the desired result.

□

References

[1] Artstein, Z., Kevrekidis, I. G., Slemrod, M. and Titi, E. S.: *Slow observables of singularly perturbed differential equations.* Nonlinearity **20** (2007), 2463-2481 (2007).

[2] Artstein, Z. and Vigodner, A.: *Singularly perturbed ordinary differential equations with dynamic limits.* Proc. Roy. Soc. Edinburgh Sec. A. **126** (1996), 541-569.

[3] Cucker, F. and Smale, S.: *On the mathematics of emergence.* Japan. J. Math. **2**, 197-227 (2007).

[4] Cucker, F. and Smale, S.: *Emergent behavior in flocks.* IEEE Trans. Automat. Control **52**, 852-862 (2007).

[5] Erdmann, U., Ebeling, W. and Mikhailov, A.: *Noise-induced transition from translational to rotational motion of swarms.* Phys. Review E **71**, 051904 (2005)

[6] Ha, S.-Y. and Tadmor, E.: *From particle to kinetic and hydrodynamic description of flocking*. Kinetic and Related Models. **1**, 415-435 (2008)
[7] Ha, S.-Y. and Levy, D.: *Particle, kinetic and fluid models for phototaxis*. Submitted.
[8] Ha, S.-Y., Lee, K. and Devy, D.: *Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system*. Submitted.
[9] Ha, S.-Y. and Liu, J.-G.: *A simple proof of Cucker-Smale flocking dynamics and mean field limit*. Submitted.
[10] O'Malley, R. E.: *Singular perturbation methods for ordinary differential equations.* Applied Mathematical Sciences, **89**. Springer-Verlag, New York, 1991.
[11] Shen, J.: *Cucker-Smale flocking under hierarchical leadership*. SIAM J. Appl. Math. **68**, 694-719 (2007/08).
[12] Tikhonov, A. N.: *Systems of differential equations containing a small parameter multiplying the derivative.* Mat. Sb. N.S. **27**, 147-156 (1950).

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