A SIMPLE PROOF OF THE CUCKER-SMALE FLOCKING DYNAMICS AND MEAN-FIELD LIMIT ∗

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Abstract. We present a simple proof on the formation of flocking to the Cucker-Smale system based on the explicit construction of a Lyapunov functional. Our results also provide a unified condition on the initial states in which the exponential convergence to flocking state will occur. For large particle systems, we give a rigorous justification for the mean-field limit from the many particle Cucker-Smale system to the Vlasov equation with flocking dissipation as the number of particles goes to infinity.

Key words. Flocking, swarming, emergence, self-driven particles system, autonomous agents, Vlasov equation, Lyapunov functional, measure valued solution, Kantorovich-Rubinstein distance.

Subject classifications. Primary 92C17; secondary 82C22, 82C40.

1. Introduction Collective self-driven synchronized motion of autonomous agents appears in many applications ranging from animal herding to the emergence of common languages in primitive societies [1, 9, 10, 11, 12, 14, 16, 18, 19, 20, 2, 3]. The word flocking in this paper refers to general phenomena where autonomous agents reach a consensus based on limited environmental information and simple rules. In the seminar work of Cucker and Smale [2, 3], they postulated a model for the flocking of birds, and verified the convergence to a consensus (the same velocity) depending on the spatial decay of the communication rate between autonomous agents.

In this paper, we present a simple and complete analysis of the flocking to the Cucker-Smale system (in short C-S system) using the explicit Lyapunov functional approach, in particular we improve the flocking estimates of the C-S system for regular and algebraically decaying communication rates [2, 3] in two ways: First, we present flocking estimates for general communication rates which can be singular when two particles are very close enough. Secondly, we remove the conditional assumption in [2, 3] on the initial configuration in critical case, where the communication rate behaves like \(|x|^{-1}\), as \(|x| \to \infty\). We show that the standard deviations of particle phase-space positions are dominated by the system of dissipative differential inequalities (SDDI):

\[
\begin{align*}
\frac{dX}{dt} &\leq V, \\
\frac{dV}{dt} &\leq -\phi(X)V,
\end{align*}
\]

(1.1)

where \((X,V)\) are nonnegative functions, and \(\phi\) is a nonnegative measurable function.

A simple phase plane analysis (see Section 3) provides a unified condition on initial states in which the convergence to a consensus will occur. For the mean-field limit for large particle systems with flocking dissipation, we derive a bounding function for the growth on the size of a velocity support. This enables us to establish stability estimate of measure valued solutions to the C-S system in Kantorovich-Rubinstein distance. For the existence of a measure valued solution, we refine the argument given in [13, 17]. The novelty of this paper is to present a direct and simple but surprisingly a complete flocking analysis employing a Lyapunov functional to the SDDI (1.1). This

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is a contrasted difference from the Cucker-Smale’s original proof [2, 3] and its modified approach in [8] using a bootstrapping argument. We refer to Degond and Motsch’s works [4, 5, 6] for recent kinetic description of Viscek type model for flocking.

The rest of the paper is divided into two main parts (flocking estimates and mean-field limit) after this introduction. The first four sections are devoted to the dissipation estimates for the SDDI system, which yield simplified and strong estimates for the dynamics of the C-S system. More precisely, in Section 2, we briefly review the C-S system and provide the reduction of Cucker-Smale dynamics to the dynamics of the SDDI. In Section 3, we present a general framework on the dissipation estimate of the SDDI using explicit Lyapunov functionals, and we also present several conditions regarding the initial configurations and communication rate functions for the flocking formation. In Section 4, we study the flocking estimates for dissipative systems with the algebraically decaying communication rates with singularity or non-singularity at the origin as a direct application of the results in Section 3, and we also briefly discuss the comparison between Cucker-Smale’s result with main results in Section 4. Finally, last two sections deal with kinetic mean-field limit of the C-S system. In Section 5, we study the particle approximation [15] to the kinetic C-S model introduced in [8]. Section 6 is devoted to the existence of measure valued solutions to the kinetic C-S model.

**Notation:** Throughout the rest part of the paper, we denote by $C(T,n,\cdots)$ be a generic positive constant depending only on $T,n$, etc., and for any vector $b$, $b^i$ represents $i$-th component of it.

2. Preliminaries

In this section, we discuss the reduction of the C-S dynamics to the SDDI (1.1) and present a definition on the time-asymptotic flocking.

Consider an interacting particle system consisting of $N$ identical autonomous agents with unit mass [2, 3, 8, 9, 16]. Let $(x_i(t),v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$ be the phase space position of $i$-particle governed by the general Cucker-Smale’s dynamical system: For $1 \leq i \leq N$,

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \lambda \frac{1}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i), \quad t > 0, \tag{2.1}
\]

with initial data

\[
x_i(0) = x_{i0}, \quad v_i(0) = v_{i0}. \tag{2.2}
\]

Here $\lambda$ and $\psi$ are a nonnegative coupling strength and the mutual communication rate between autonomous agents respectively.

Note that the vector field $(v, \lambda F)$, $F^i = \frac{1}{N} \sum_{j=1}^{N} \psi(|x_j - x_i|)(v_j - v_i)$ associated with (2.1) satisfies dissipative condition:

\[
\nabla(x,v) \cdot (v, \lambda F) = \lambda \sum_{i=1}^{N} \nabla v_i \cdot F^i = -\lambda \frac{1}{N} \sum_{i,j=1}^{N} \psi(|x_j - x_i|) \leq 0.
\]

This implies that the C-S model (2.1) is a dissipative dynamical system. We first set the center of mass system $(x_c,v_c)$:

\[
x_c := \frac{1}{N} \sum_{k=1}^{N} x_k, \quad v_c := \frac{1}{N} \sum_{k=1}^{N} v_k. \tag{2.3}
\]
Then the system (2.1) implies
\[ \frac{dx_c}{dt} = v_c, \quad \frac{dv_c}{dt} = 0, \]
which gives explicit solution
\[ x_c(t) = x_c(0) + tv_c(0), \quad v_c(t) = v_c(0). \]
Without loss of generality, we may assume that the center of mass coordinate of the system is fixed at zero in phase space at time \( t \):
\[ x_c(t) = 0, \quad v_c(t) = 0, \quad t \geq 0. \tag{2.4} \]
which is equivalent to the relations:
\[ \sum_{i=1}^{N} x_i(t) = 0, \quad \sum_{i=1}^{N} v_i(t) = 0, \quad t \geq 0. \tag{2.5} \]

If necessary, instead of \((x_i, v_i)\) we may consider new variables \((\hat{x}_i, \hat{v}_i) := (x_i - x_c, v_i - v_c)\) which correspond to the fluctuations around the center of mass system.

### 2.1. Reduction of the C-S dynamics to the SDDI’s dynamics

In this part, we explain how the dynamics of the C-S system (2.1) with (2.4) can be determined by the corresponding dynamics of the SDDI, which will be discussed in next section. We now consider the system (2.1) with (2.4). It follows from (2.1) that we have
\[ \frac{d}{dt} \sum_{i=1}^{N} ||v_i||^2 = -\lambda \frac{N}{N} \sum_{1 \leq i, j \leq N} \psi(|x_j - x_i|)||v_j - v_i||^2. \tag{2.6} \]

Here \( ||\cdot|| \) is the standard \( l_2 \)-norm in \( \mathbb{R}^d \).

We set
\[ x := (x_1, \cdots, x_N) \in \mathbb{R}^{Nd}, \quad v := (v_1, \cdots, v_N) \in \mathbb{R}^{Nd}, \]
and
\[ ||x|| = \left( \sum_{i=1}^{N} ||x_i||^2 \right)^{\frac{1}{2}}, \quad ||v|| = \left( \sum_{i=1}^{N} ||v_i||^2 \right)^{\frac{1}{2}}. \]

Note that \( ||x|| \) and \( ||v|| \) denote quantities proportional to the standard deviations of \( x_i \) and \( v_i \) respectively.

**Lemma 2.1.** Let \((x_i, v_i)\) be the solution to (2.1) with a nonnegative and non-increasing function \( \psi \). Assume that (2.4) holds. Then \( ||x|| \) and \( ||v|| \) satisfy the SDDI (1.1) with \( \phi(s) = \psi(2s) \):
\[ \frac{d||x||}{dt} \leq ||v||, \quad \frac{d||v||}{dt} \leq -\frac{\lambda}{N} \psi(2||x||)||v||. \tag{2.7} \]

**Proof.** We take an inner product the C-S system (2.1) with \( 2x_i \) and use Cauchy-Schwartz inequality to see
\[ \frac{d||x||^2}{dt} = 2\frac{dx}{dt},x = 2(v, x) \leq 2||x||||v||, \]
where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^{Nd} \). This gives the first inequality in (2.7). Next, we use (2.6) and the nonnegativity and non-increasing properties of \( \psi \), and the fact

\[
\max_{1 \leq i, j \leq N} |x_i - x_j| \leq 2\|x\|,
\]

to find

\[
\frac{d\|v\|^2}{dt} \leq -\frac{\lambda}{N} \psi(2\|x\|) \sum_{1 \leq i, j \leq N} \|v_j - v_i\|^2 = -\frac{2\lambda}{N} \psi(2\|x\|)\|v\|^2.
\]

This gives the second inequality in (2.7). Here we used (2.5) to see

\[
\sum_{1 \leq i, j \leq N} \|v_i - v_j\|^2 = 2\sum_{i=1}^N \|v_i\|^2 - 2 \left\langle \sum_{i=1}^N v_i, \sum_{j=1}^N v_j \right\rangle = 2\|v\|^2.
\]

\[\square\]

The above reduction to the SDDI can be recast in a more abstract form. Let \( E \) be a vector space over \( \mathbb{R} \) with an inner product \( \langle \cdot, \cdot \rangle \) and its corresponding norm \( \|\cdot\| \). Let \( (x_i, v_i) \in \mathbb{R}^{2d} \) be the phase space coordinate of \( i \)-th autonomous agents among \( N \) agents. In this case, \( E \) is simply the \( N \)-particle phase space \( \mathbb{R}^{Nd} \). Consider the following dynamical system in \( E \times E \):

\[
dx{t} = v, \quad dv{t} = -L(x)v, \quad t > 0,
\]

with initial data

\[
x(0) = x_0, \quad v(0) = v_0.
\]

where \( L(x):E \rightarrow E \) is a linear operator. We assume that the linear operator \( L(x) \) satisfies a coercivity condition: There is a positive and non-increasing function \( \psi(s) \) such that

\[
\langle L(x)v, v \rangle \geq \psi(\|x\|)\|v\|^2.
\]

Here \( \psi \) is a nonnegative measurable function. We set \( \Psi \) to be the primitive function of \( \psi \):

\[
\Psi'(s) = \psi(s), \quad s \geq 0.
\]

Then it is easy to see that \( \Psi \) is an increasing function. We next show that \((\|x\|, \|v\|)\) to the system (2.8) with (2.10) satisfies the SDDI. It follows from (2.8) that

\[
\frac{d\|x\|^2}{dt} = 2\langle x, v \rangle \leq 2\|x\|\|v\|,
\]

\[
\frac{d\|v\|^2}{dt} = -2\langle L(x)v, v \rangle \leq -2\psi(\|x\|)\|v\|^2.
\]

These yield the SDDI:

\[
\frac{d\|x\|}{dt} \leq \|v\|, \quad \frac{d\|v\|}{dt} \leq -\psi(\|x\|)\|v\|.
\]

(2.11)
Before we close this section, we present the definition of the time-asymptotic
flocking as follows.

**Definition 2.2.** The Cucker-Smale system (2.1) has a time-asymptotic flocking if and only if the solutions \( \{x_i, v_i\}, i = 1, \ldots, N \) to (2.1) satisfy the following two conditions:

1. The velocity fluctuations go to zero time-asymptotically (velocity alignment):
\[
\lim_{t \to +\infty} \sum_{i=1}^{N} ||v_i(t) - v_c(t)||^2 = 0.
\]

2. The position fluctuations are uniformly bounded in time \( t \) (forming a group):
\[
\sup_{0 \leq t < \infty} \sum_{i=1}^{N} ||x_i(t) - x_c(t)||^2 < \infty.
\]

3. A flocking theorem

Note that the SDDI (2.11) admits natural Lyapunov functionals \( E_{\pm}(x, v) \) which can be viewed as energy functionals for (2.11):
\[
E_{\pm}(||x||, ||v||) := ||v|| \pm \Psi(||x||).
\]

The first and second terms in the \( E_{\pm}(||x||, ||v||) \) can be regarded as the kinetic and internal (potential) energies respectively. Next lemma shows that the functionals \( E_{\pm}(||x||, ||v||) \) are non-increasing along the solutions \( (||x||, ||v||) \) of (2.11), which means the dissipativity of the system (2.8)

**Lemma 3.1.** Suppose \( (||x||, ||v||) \) satisfy the SDDI (2.11) with \( \psi \geq 0 \). Then we have

\( i \) \( E_{\pm}(||x(t)||, ||v(t)||) \leq E_{\pm}(||x_0||, ||v_0||), \ t \geq 0. \)

\( ii \) \( ||v(t)|| + \int_{||x_0||}^{||x(t)||} \psi(s)ds \leq ||v_0||. \)

**Proof.** (i) We now use (2.11) to get
\[
\frac{d}{dt} E_{\pm}(||x(t)||, ||v(t)||) = \frac{d}{dt} \left( ||v(t)|| \pm \Psi(||x(t)||) \right)
\]
\[
= \frac{d}{dt} ||v|| \pm \psi(||x||) \frac{d}{dt} ||x||
\]
\[
\leq \psi(||x||) \left( -||v|| \pm \frac{d}{dt} ||x|| \right)
\]
\[
\leq 0.
\]

(ii) It follows from the result (i) that
\[
||v(t)|| - ||v_0|| \leq -\left( \Psi(||x(t)||) - \Psi(||x_0||) \right), \ ||v(t)|| - ||v_0|| \leq (\Psi(||x(t)||) - \Psi(||x_0||)).
\]

Hence we have
\[
||v(t)|| - ||v_0|| \leq -\left| \Psi(||x(t)||) - \Psi(||x_0||) \right| = -\int_{||x_0||}^{||x(t)||} \psi(s)ds.
\]
\[\square\]
As a direct application of Lemma 3.1, we obtain the following bounds on \( |x| \) and \( |v| \) satisfying the SDDI.

**Theorem 3.2.** Suppose \((|x|, |v|)\) satisfy the SDDI (2.11) with \( \psi \geq 0 \). Then the following estimates hold.

(i) If

\[
\|v_0\| < \int_0^\infty \|x_0\| \psi(s) ds,
\]

then there is an \( x_m \geq 0 \) such that

\[
\|v_0\| = \int_{x_m}^{\|x_0\|} \psi(s) ds, \quad \|x(t)\| \geq x_m, \quad t \geq 0.
\]

(ii) If

\[
\|v_0\| < \int_{\|x_0\|}^\infty \psi(s) ds,
\]

then there is an \( x_M \geq 0 \) such that

\[
\|v_0\| = \int_{\|x_0\|}^{x_M} \psi(s) ds, \quad \|x(t)\| \leq x_M, \quad \|v(t)\| \leq \|v_0\| e^{-\psi(x_M)t}.
\]

**Proof.** (i) Since \( \psi \) is a nonnegative measurable function,

\[
\int_0^\delta \psi(s) ds \quad \text{is a non-increasing continuous function in } \delta \geq 0.
\]

Hence if \( \|v_0\| < \int_0^\|x_0\| \psi(s) ds \), let \( x_m > 0 \) to be the smallest value such that

\[
\|v_0\| = \int_{x_m}^{\|x_0\|} \psi(s) ds.
\]

(3.2)

For the second assertion, we use the result (ii) in Lemma 3.1, i.e., for any solution \((x, v)\) to (2.8), we have

\[
\left| \int_{\|x_0\|}^{|x(t)|} \psi(s) ds \right| \leq \|v_0\|, \quad t \geq 0.
\]

(3.3)

Suppose there exists a \( t \in (0, \infty) \) such that

\[
\|x(t)\| < x_m.
\]

Then we can choose a time \( t_* \in (0, \infty) \) such that

\[
\|v_0\| < \int_{\|x(t_*)|}^{\|x_0\|} \psi(s) ds, \quad \|x(t_*)| < x_m, \quad \min_{0 \leq s \leq t_*} \|x(s)\| \geq \frac{x_m}{2}.
\]

We now consider \( \Psi \) defined as follows.

\[
\Psi(s) := \int_{\|x_0\|}^s \psi(\tau) d\tau.
\]

(3.4)
Then for such $t_*$ and $\Psi$, we have
\[
\|v_0\| = \int_{x_m} \|x_0\| \psi(s) ds < \int \|x(t_*)\| \psi(s) ds,
\]
which is contradictory to (3.3).

(ii) We use the same argument as (i), i.e.,
\[
\mbox{If } \|v_0\| < \int \|x_0\| \psi(s) ds,
\]
then we choose the largest value $x_M \geq 0$ to satisfy
\[
\|v_0\| := \int_{x_M}^{x_0} \psi(s) ds.
\]
Similar to (i), it is easy to see that
\[
\|x(t)\| \leq x_M.
\]
On the other hand, we use the above upper bound for $||x(t)||$ to find
\[
\frac{d\|v\|}{dt} \leq -\psi(x_M) \|v\|.
\]
This yields the decay estimate for $||v(t)||$:
\[
||v(t)|| \leq ||v_0|| e^{-\psi(x_M)t}.
\]

In next section, we consider the SDDI (2.11) equipped with explicit singular and regular $\psi$'s decaying algebraically at infinity, and apply Theorem 3.1 to get the detailed information on the size of spatial support and time-decay estimates of $||v||$.

4. Application of the flocking theorem to the Cucker-Smale system

In this section, we consider two explicit communication rate functions $\psi$:
\[
\psi_1(s) := \frac{\alpha s^\beta}{s^\gamma}, \quad \psi_2(s) := \frac{\alpha}{(1+s^2)^{\frac{\beta}{2}}}, \quad \alpha > 0, \quad \beta \geq 0.
\]
The second communication rate function $\psi_2$ has been employed in previous literatures [2, 3, 8, 16] on flocking. In the following two subsections, we present several estimates for $x_m$ and $x_M$ together with explicit dissipation estimates.

4.1. Singular communication rate

In this part, we consider the singular $\psi$ decaying algebraically at infinity:
\[
\psi_1(s) = \frac{\alpha}{s^\beta}, \quad \alpha > 0, \quad \beta \geq 0.
\]
Since the flocking dynamics of $\psi_1$ completely depends on $\beta \in [0,1]$ or $\beta \in (1,\infty)$, we separate its presentation. For the long-range communication rate $\psi_1$ with $\beta \in [0,1]$, we have global flocking regardless of initial configurations. In contrast, for the case of short range case with $\beta \in (1,\infty)$, we have a conditional flocking saying that only certain classes of initial configurations reduce to the flocking state.
Proposition 4.1. (Unconditional flocking) Suppose \((||x||,||v||)\) satisfy the SDDI (2.11) with \(\psi = \psi_1\):

\[
\beta \in [0,1], \quad ||x_0|| \neq 0.
\]

Then there exist \(x_m\) and \(x_M\) independent of \(t\) satisfying

\[
x_m \leq ||x(t)|| \leq x_M, \quad ||v(t)|| \leq ||v_0||e^{-\psi(x_M)t},
\]

where \(x_m\) and \(x_M\) are explicitly given by

\[
x_m := \left\{ \begin{array}{ll}
\left( \max \left\{ 0, ||x_0||^{1-\beta} - \frac{1-\beta}{\alpha} ||v_0|| \right\} \right)^{1/\beta}, & \beta \in [0,1), \\
||x_0||e^{-\frac{||v_0||}{||x_0||}}, & \beta = 1,
\end{array} \right. \quad \text{and}
\]

\[
x_M := \left\{ \begin{array}{ll}
\left( ||x_0||^{1-\beta} + \frac{1-\beta}{\alpha} ||v_0|| \right)^{1/\beta}, & \beta \in [0,1), \\
||x_0||e^{-\frac{||v_0||}{||x_0||}}, & \beta = 1.
\end{array} \right.
\]

Proof. If \(||v_0|| = 0\), then we have \(||v(t)|| = 0\) and \(||x(t)|| = ||x_0||\). So the lemma holds. Now we show the lemma also holds for the case of \(||v_0|| \neq 0\).

(i) (Estimates of \(x_M\)): For \(\beta \in [0,1]\), since

\[
\int_{||x_0||}^{\infty} \psi_1(s)ds = \int_{||x_0||}^{\infty} \frac{\alpha}{s^\beta}ds = \infty,
\]

it follows from Theorem 3.2 that there exists a \(x_M > 0\) such that

\[
||v_0|| = \int_{||x_0||}^{x_M} \psi_1(s)ds, \quad ||x(t)|| \leq x_M, \quad ||v(t)|| \leq ||v_0||e^{-\psi(x_M)t}.
\]

Below, we give a upper bound for \(x_M\) using the above defining relation.

Case 1 \((0 \leq \beta < 1)\): By direct calculation, we have

\[
||v_0|| = \int_{||x_0||}^{x_M} \psi(s)ds = \frac{\alpha}{1-\beta}(x_M^{1-\beta} - ||x_0||^{1-\beta}), \quad \text{i.e.}
\]

\[
x_M = \left( ||x_0||^{1-\beta} + \frac{1-\beta}{\alpha} ||v_0|| \right)^{1/\beta}.
\]

Case 2 \((\beta = 1)\): In this case, we have

\[
||v_0|| = \int_{||x_0||}^{x_M} \psi_1(s)ds = \alpha \ln \frac{x_M}{||x_0||}, \quad \text{i.e.,} \quad x_M = ||x_0||e^{\frac{||v_0||}{\alpha}}.
\]

(ii) (Estimates of \(x_m\)): As in (i), we separate the estimate into two cases.

Case 1 \((0 \leq \beta < 1)\): We again use Theorem 3.2 to get the estimate of \(x_m\). Note that

\[
\int_{0}^{||x_0||} \psi_1(s)ds = \frac{\alpha}{1-\beta}||x_0||^{1-\beta}, \quad \int_{x_m}^{||x_0||} \psi_1(s)ds = \frac{\alpha}{1-\beta}(||x_0||^{1-\beta} - x_m^{1-\beta}).
\]
If \( \|v_0\| < \alpha \beta^{-\alpha/\beta} \|x_0\|^{1-\beta} \), then the defining condition for \( x_m \) yields the explicit representation for \( x_m \):

\[
\|v_0\| = \int_{x_m}^{\|x_0\|} \psi_1(s) ds, \quad \text{i.e.} \quad \|v_0\| = \frac{\alpha}{1-\beta} (\|x_0\|^{1-\beta} - x_m^{1-\beta}).
\]

Hence we have

\[
x_m = \left( \|x_0\|^{1-\beta} - \frac{1-\beta}{\alpha} \|v_0\| \right)^{\frac{1}{1-\beta}}.
\]

Otherwise, we simply take \( x_m = 0 \).

**Case 2 (\( \beta = 1 \))**: Since

\[
\int_0^{\|x_0\|} \psi_1(s) ds = \infty,
\]

we use Theorem 3.2 directly to get

\[
\|v_0\| = \int_{x_m}^{\|x_0\|} \frac{\alpha}{s} ds, \quad x_m = \|x_0\| e^{-\psi(x_M) t}.
\]

\( \square \)

Below, we consider the case \( \beta > 1 \), which is the integrable case at \( s = \infty \).

**Proposition 4.2.** (Conditional flocking) Suppose \((\|x\|,\|v\|) \) satisfy the SDDI (2.11) with \( \psi = \psi_1 \):

\[
\beta \in (1, \infty), \quad \|x_0\| \neq 0, \quad \|x_0\|^{1-\beta} > \frac{\beta-1}{\alpha} \|v_0\|.
\]

Then there exist \( x_m \) and \( x_M \) independent of \( t \) satisfying

\[
x_m \leq \|x(t)\| \leq x_M, \quad \|v(t)\| \leq \|v_0\| e^{-\psi(x_M) t},
\]

where \( x_m \) and \( x_M \) are given by

\[
x_m := \left( \|x_0\|^{1-\beta} + \frac{\beta-1}{\alpha} \|v_0\| \right)^{\frac{1}{1-\beta}}, \quad x_M := \left( \|x_0\|^{1-\beta} - \frac{\beta-1}{\alpha} \|v_0\| \right)^{\frac{1}{1-\beta}}.
\]

**Proof.** As in the prove of Proposition 4.1, the lemma holds for \( \|v_0\| = 0 \). So we only need to prove the lemma for the case of \( \|v_0\| \neq 0 \).

(i) Note that

\[
\int_0^{\infty} \frac{\alpha}{s^\beta} ds = \frac{\alpha}{\beta-1} \|x_0\|^{1-\beta} < \infty.
\]

We again use Theorem 3.2 to find \( x_M \): If

\[
\|v_0\| < \frac{\alpha}{\beta-1} \|x_0\|^{1-\beta},
\]
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then

\[ \|v_0\| = \int_{\|x_0\|}^{x_M} \frac{\alpha}{s^\beta} ds = \frac{\alpha}{\beta - 1} (\|x_0\|^{1-\beta} - x_M^{1-\beta}), \]

which gives the desired result.

(ii) Since

\[ \int_0^{\|x_0\|} \frac{\alpha}{s^\beta} ds = \infty, \]

we can find \(x_m\) from the defining relation:

\[ \|v_0\| = \int_{x_m}^{\|x_0\|} \frac{\alpha}{s^\beta} ds = \frac{\alpha}{\beta - 1} (x_m^{1-\beta} - \|x_0\|^{1-\beta}). \]

\(\Box\)

4.2. Regular communication rate

In this part, we will consider regular force case at \(s = 0\). For definiteness, we set

\[ \psi_2(s) := \frac{\alpha}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \alpha > 0, \quad \beta \geq 0. \]

In the following, we will use the fact that

\[ \int_{a}^{b} (1 + s^2)^{-\frac{\beta}{2}} ds = \int_{a}^{b} s(1 + s^2)^{-\frac{\beta+1}{2}} ds = \frac{1}{1 - \beta} (1 + s^2)^{1-\beta} |_{a}^{b}, \quad \beta \neq 1, \]

\[ \int_{a}^{b} (1 + s^2)^{-\frac{\beta}{2}} ds = \ln (s + \sqrt{1 + s^2}) |_{a}^{b}, \quad \beta = 1. \]

Below, we present two parallel propositions with Proposition 4.1 and 4.2 without proofs.

**Proposition 4.3.** (Unconditional flocking) Suppose \((\|\vec{x}\|, \|\vec{v}\|)\) satisfy the SDDI (2.11) with \(\psi = \psi_2; \beta \in [0,1].\)

Then there exist \(x_m\) and \(x_M\) independent of \(t\) satisfying

\[ x_m \leq \|x(t)\| \leq x_M, \quad \|v(t)\| \leq \|v_0\| e^{-\psi(x_M)t}, \]

where \(x_m\) and \(x_M\) are given by the solution of the following relations:

\[ \begin{cases} 
(1 + x_m^2)^{\frac{1-\beta}{2}} \geq \max \left\{ 1, (1 + \|x_0\|^2)^{\frac{1-\beta}{2}} - \frac{1-\beta}{\alpha} \|v_0\| \right\}, & \beta \in [0,1), \\
(1 + \|x_0\|^2)^{\frac{1-\beta}{2}} \geq \max \left\{ 1, (\|x_0\| + \sqrt{1 + \|x_0\|^2}) e^{-\|v_0\|/\alpha} \right\}, & \beta = 1. 
\end{cases} \]

and

\[ \begin{cases} 
(1 + x_M^2)^{\frac{1-\beta}{2}} \leq (1 + \|x_0\|^2)^{\frac{1-\beta}{2}} + \frac{1-\beta}{\alpha} \|v_0\|, & \beta \in [0,1), \\
x_M + \sqrt{1 + x_M^2} = (\|x_0\| + \sqrt{1 + \|x_0\|^2}) e^{\frac{\|v_0\|}{\alpha}}, & \beta = 1, 
\end{cases} \]
Proof. The proof is similar to that of Proposition 4.1. Hence we omit its detailed proof. □

Proposition 4.4. (Conditional flocking) Suppose \((|x|, |v|)\) satisfy the SDDI (2.11) with \(\psi = \psi_2\):

\[
\beta \in (1, \infty), \quad (1 + \|x_0\|^2)^{\frac{\alpha}{\alpha-1}} > \frac{(\beta - 1)}{\alpha} \|v_0\|.
\]

Then there exist \(x_m\) and \(x_M\) independent of \(t\) satisfying

\[
x_m \leq \|\hat{x}(t)\| \leq x_M, \quad \|\hat{v}(t)\| \leq \|\hat{v}_0\| e^{-\psi(x_M)t},
\]

where \(x_m\) and \(x_M\) are given by the solution of the following relations:

\[
(1 + x_m^2)^{\frac{\alpha-1}{\alpha}} \geq \left[ \min \left\{ 1, (1 + \|x_0\|^2)^{\frac{\alpha}{\alpha-1}} + \frac{(\beta - 1)}{\alpha} \|v_0\| \right\} \right]^{-1},
\]

\[
(1 + x_M^2)^{\frac{\alpha-1}{\alpha}} \leq \left[ (1 + \|x_0\|^2)^{\frac{\alpha}{\alpha-1}} - \frac{(\beta - 1)}{\alpha} \|v_0\| \right]^{-1}.
\]

4.3. Comparison with Cucker-Smale’s results  

In this part, we briefly summarize improved flocking estimates on the C-S model (2.1) with \(\psi_i, i = 1, 2\), and compare our results with these given in [2, 3] for the communication rate \(\psi_i\). Since the C-S model can be reduced to the SDDI (see Section 2.1), we can use the estimates in Proposition 4.3 and 4.4. The main results on flocking phenomena can be summarized as follows.

Theorem 4.5. (Unconditional flocking) Assume that the communication rate is \(\psi = \psi_i, i = 1, 2\) with \(\beta \in [0, 1]\). Let \((x, v)\) be a solution to (2.1) with \(x_0 \neq 0\). Then there exist positive constants \(x_m\) and \(x_M\) independent of \(t\) satisfying

\[
x_m \leq \|x(t)\| \leq x_M, \quad \|v(t)\| \leq \|v_0\| e^{-\psi(x_M)t}, \quad t \geq 0.
\]

Remark 4.1. The results given in the above theorem improves the flocking results in Cucker-Smale [2, 3, 8, 16] which only deal with regular communication rate \(\psi_2\) in two aspects: Improved flocking estimates for critical case \((\beta = 1)\) and singular communication rate \(\psi_1\). More precisely,

(i) For \(0 \leq \beta < 1\), unconditional flocking with exponential decay in the variance of velocity was obtained both in [2] and [3].

(ii) For the system (2.1) with regular communication rate \(\psi_2\) with \(\beta = 1\), conditional flocking for initial configuration satisfying

\[
\|\hat{v}_0\|^2 \leq C(N, \lambda),
\]

was obtained in [2, 3], whereas Ha-Tadmor [8] improved Cucker-Smale’s conditional flocking for the unconditional flocking with algebraic decay rates. Hence the result given in this theorem also improves the results [2, 3, 8].
Theorem 4.6. (Conditional flocking) Let \((x, v)\) be a solution to (2.1) with \(x_0 \neq 0\), and assume that \(\psi\) takes one of the form \(\psi_i\) with \(\beta \in (1, \infty)\). Suppose the initial configuration \((x_0, v_0)\) satisfies
\[
\left\{
\begin{aligned}
\|x_0\|^{1-\beta} > \frac{\beta-1}{\beta} \|v_0\|, & \quad \psi = \psi_1, \\
(1 + \|x_0\|^2)^{\frac{1}{2}} > \frac{\beta-1}{\alpha} \|v_0\|, & \quad \psi = \psi_2.
\end{aligned}
\right.
\]
Then there exist positive constants \(x_m\) and \(x_M\) independent of \(t\) satisfying
\[
x_m \leq \|\hat{x}(t)\| \leq x_M, \quad \||\hat{v}(t)\| \leq \|\hat{v}_0\| e^{-\psi(x_M)t}, \quad t \geq 0.
\]

Remark 4.2. For the short-range communication rate \(\psi_2\) with \(\beta > 1\), i.e., in the Cucker-Smale’s context \(K = \lambda N, \sigma = 1\), the sufficient condition for flocking formation given in [2, 3] can be rephrased as follows.
\[
A \pi^{\frac{1}{\beta}} \left[ \left( \frac{1}{\beta} \right)^{\frac{1}{1-\beta}} - \left( \frac{1}{\beta} \right)^{\frac{2}{\beta}} \right] > B,
\]
where
\[
A = \frac{\nu^2 \lambda^2}{8 N^3 \|v_0\|^2}, \quad B = 1 + 2N \|x_0\|^2,
\]
which can be recast as
\[
(1 + 2N \|x_0\|^2)^{\frac{1}{1-\beta}} > \frac{(2N)^2 \|v_0\|}{\nu \lambda} \left[ \left( \frac{1}{\beta} \right)^{\frac{1}{1-\beta}} - \left( \frac{1}{\beta} \right)^{\frac{2}{\beta}} \right] \frac{1}{\beta}.
\]
Here \(\nu = \nu(N)\) is a positive constant bounded by
\[
\frac{1}{3N} \leq \nu(N) \leq 2N(N-1), \quad \text{for } N \geq 2.
\]

The discrepancy between condition in Theorem 4.6 and (4.1) might be due to different a priori estimates.

5. Measure valued solution and stability estimate In this section, we present a global existence of measure valued solutions for the kinetic C-S model [8] and their stability estimate in Kantorovich-Rubinstein distance. Recall C-S particle model: For \(i = 1, \ldots, N\),
\[
\begin{cases}
\frac{dx_i}{dt} = v_i, & \\
\frac{dv_i}{dt} = \lambda \sum_{j=1}^{N} m_j \psi(|x_j - x_i|)(v_j - v_i), & m_j : \text{constant},
\end{cases}
\]
and the corresponding kinetic C-S model:
\[
\begin{cases}
\partial_t f + \text{div}_x (vf) + \lambda \text{div}_v (F[f]) = 0, \\
f(x, v, 0) = f_0(x, v),
\end{cases}
\]
where \( f \) is the one-particle distribution function, and \( F[f] \) is given by the following representation:

\[
F[f](x,v,t) := -\int_{\mathbb{R}^d} \psi(|x-y|)(v-v_*)f(y,v_*,t)dv_*dx.
\]

The kinetic model (5.2) was introduced by Ha and Tadmor [8] using the method of BBGKY hierarchy from the C-S particle model as a mesoscopic description for flocking.

**5.1. Measure valued solution and moment estimates** In this part, we review the notion of a measure valued solution to (5.2) and present several a priori estimates for the particle trajectory. In the time-asymptotic limit \((t \to \infty)\), the distribution function \( f = f(x,v,t) \) will concentrate on the velocity mean value (see [2, 3, 8]), hence the natural solution space for the kinetic equation (5.2) will be the space of nonnegative measures including Dirac measures. We will prove that the kinetic C-S model (5.2) is well-posed in the space of Radon measures.

Let \( \mathcal{M}(\mathbb{R}^d) \) be the set of nonnegative Radon measures on the phase space \( \mathbb{R}^d \), which can be understood as nonnegative bounded linear functionals on \( C_0(\mathbb{R}^d) \). For a Radon measure \( \nu \in \mathcal{M}(\mathbb{R}^d) \), we use a standard duality relation:

\[
\langle \nu, g \rangle := \int_{\mathbb{R}^d} g(x,v)\nu(dx,dv), \quad g \in C_0(\mathbb{R}^d).
\]

The definition of a measure-valued solution to the equation (5.2) is given as follows.

**Definition 5.1.** For \( T \in [0, \infty) \), let \( \mu \in L^\infty([0,T]; \mathcal{M}(\mathbb{R}^d)) \) be a measure valued solution to (5.2) with initial Radon measure \( \mu_0 \in \mathcal{M}(\mathbb{R}^d) \) if and only if \( \mu \) satisfies the following conditions:

1. \( \mu \) is weakly continuous:
   \[
   \langle \mu_t, g \rangle \text{ is continuous as a function of } t, \quad \forall \ g \in C_0(\mathbb{R}^d).
   \]

2. \( \mu \) satisfies the integral equation:
   \[
   \langle \mu_t, g(\cdot, t) \rangle - \langle \mu_0, g(\cdot, 0) \rangle = \int_0^t \langle \mu_s, \partial_s g + v \cdot \nabla_x g + \lambda F(x,v) g \rangle ds, \tag{5.3}
   \]
   where \( F(x,v)g \) is a forcing term defined as follows:

\[
F(x,v)g := -\int_{\mathbb{R}^d} \psi(|x-y|)(v-v_*)\mu_s(dy,dv_*).
\]

**Remark 5.1.** 1. If \( f \in L^1(\mathbb{R}^d \times [0,T]) \) is a weak solution (in the sense of a distribution) to the (5.2). Then \( \mu_t(dx,dv) = f(x,v,t)dx dv \) is a measure valued solution to (5.2).

2. If \( \mu \) is a measure valued solution to (5.2) and \( \mu_t \) be the absolutely continuous measure with respect to Lebesgue measure whose distribution function is given by \( f \in L^1(\mathbb{R}^d \times [0,T]) \), i.e., \( \mu_t(dx,dv) = f(x,v,t)dx dv \), then \( f \) is a distributional weak solution to the (5.2).

3. For any solutions \((x_i,v_i)\) to particle system (5.1), the discrete measure
\[ \mu_t := \sum_{i=1}^{N} m_i \delta(x - x_i(t)) \otimes \delta(v - v_i(t)), \quad m_i : \text{constant}, \]
is a measure valued solution.

4. Recall that \( \text{supp}(\mu) \) (the support of a measure \( \mu \)) is the closure of the set consisting of all points \( (x, v) \) in \( \mathbb{R}^{2d} \) such that \( \mu(B_r((x, v))) > 0, \quad \forall \ r > 0. \) For a finite measure with a compact support, we can use \( g \in C^1(\mathbb{R}^{2d}) \) as a test function in (5.3).

5. Let \( \mu \) be a measure valued solution to (5.2) with initial Radon measure \( \mu_0 \) (see Section 6 for existence issue), then for any \( g \in C^1_0(\mathbb{R}^{2d}) \), we have
\[
\frac{d}{dt}(\mu_t, g) = \langle \mu_t, v \cdot \nabla_x g \rangle + \lambda \langle \mu_t, F \cdot v \rangle,
\]
as long as the L.H.S. of the above relation is well-defined. In fact, for the measure valued solution \( \mu \) in Theorem 6.2, \( \frac{d}{dt}(\mu_t, g) \) exists a.e.

6. We will revisit the formulation (5.3) from the viewpoint of particle trajectory in Lemma 5.5.

**Lemma 5.2.** Let \( \mu_t \in L^\infty((0,T);\mathcal{M}(\mathbb{R}^{2d})) \) be a nonnegative measure valued solution to (5.2) such that \( \mu_t \) has a compact support for \( t \text{ a.e.} \) Then we have
\[
\begin{align*}
(i) \quad & \frac{d}{dt} \int_{\mathbb{R}^{2d}} \mu_t(dx, dv) = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^{2d}} v \mu_t(dx, dv) = 0. \\
(ii) \quad & \frac{d}{dt} \int_{\mathbb{R}^{2d}} \langle |v|^2 \mu_t, dv \rangle = -\int_{\mathbb{R}^{2d}} \left[ \int_{\mathbb{R}^{2d}} \psi(|x-y|)|v - v_*|^2 \mu_t(dy, dv_*) \right] \mu_t(dx, dv).
\end{align*}
\]

**Proof.** The time derivative of velocity moments can be directly checked by taking \( g(x,v,t) = 1, v^i, |v|^2 \) in (5.3) respectively (see Remark 5.1 (4)), i.e.
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} v^i \mu_t(dx, dv) &= \lambda \langle \mu_t, F \cdot v v^i \rangle = \lambda \langle \mu_t, F^i \rangle \\
&= -\lambda \int_{\mathbb{R}^{2d}} \psi(|x-y|)(v^i - v_*^i) \mu_t(dy, dv_*) \mu_t(dx, dv) = 0,
\end{align*}
\]
and
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^{2d}} \langle |v|^2 \mu_t, dv \rangle &= 2\lambda \langle \mu_t, F \cdot v \rangle \\
&= -2\lambda \int_{\mathbb{R}^{2d}} \psi(|x-y|) v \cdot (v - v_*) \mu_t(dy, dv_*) \mu_t(dx, dv) \\
&= -\lambda \int_{\mathbb{R}^{2d}} \psi(|x-y|) |v - v_*|^2 \mu_t(dy, dv_*) \mu_t(dx, dv).
\end{align*}
\]

Here we used the change of variables \( (x,v) \leftrightarrow (y,v_*) \) and Fubini’s theorem. \( \square \)

5.2. **A priori estimates on the particle trajectory**

In this part, we present several a priori estimates for the particle trajectory which are crucial in next section on the existence of a measure valued solution and the mean-field limit.

We first note that the forcing term \( F(x,v,\mu_t) \) in (5.4) can be rewritten as
\[
F(x,v,\mu_t) = -\int_{\mathbb{R}^{2d}} \psi(|x-y|)(v - v_*) \mu_t(dy, dv_*) \\
= \left[ \int_{\mathbb{R}^{2d}} \psi(|x-y|) v_\mu t(dy, dv_*) \right] - \left[ \int_{\mathbb{R}^{2d}} \psi(|x-y|) \mu_t(dy, dv_*) \right] v \tag{5.5}
:= a(x, \mu_t) - b(x, \mu_t)v.
\]
Lemma 5.3. Assume the communication rate $\psi$ takes the form of
\[ \psi(s) = \frac{1}{(1 + s^2)^{\frac{\beta}{2}}} \quad \beta \geq 0, \]
and let $\mu \in L^\infty([0,T];\mathcal{M}(\mathbb{R}^d))$ be a measure valued function with the properties:
1. A compact support for each time slice: For some nonnegative locally bounded functions $R(t), P(t)$,
   \[ \text{spt}(\mu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0). \]
2. Uniform boundedness of the first two moments:
   \[ \int_{\mathbb{R}^d} \mu_t(dx, dv) \leq m_0 < \infty, \quad \int_{\mathbb{R}^d} |v|^2 \mu_t(dx, dv) \leq m_2 < \infty. \]

Then $a(x, \mu_t)$ and $b(x, \mu_t)$ satisfy
(i) $|a(x, \mu_t)| \leq (m_0 m_2)^{\frac{1}{2}}$, \quad $|a(x, \mu_t) - a(y, \mu_t)| \leq \beta |x - y|(m_0 m_2)^{\frac{1}{2}}$.
(ii) $|b(x, \mu_t)| \leq m_0$, \quad $|b(x, \mu_t) - b(y, \mu_t)| \leq \beta |x - y|$.
(iii) $|F(x, v, \mu_t) - F(y, v_*, \mu_t)| \leq \beta \left( (m_0 m_2)^{\frac{1}{2}} + P(t) + m_0 \right) |(x,v) - (y,v_*)|.$

Here $B_r(0)$ denotes the ball with a radius $r$ and a center $0$.

Proof. Let $(x,v) \in \text{spt}(\mu_t)$.

(i) We use $|\psi| \leq 1$ and Lemma 5.2 to get
\[ |a(x, \mu_t)| \leq \int_{\mathbb{R}^d} \psi(|x - y|) |v_*| \mu_t(dy, dv_*) \]
\[ \leq \left( \int_{\mathbb{R}^d} \mu_t(dy, dv_*) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |v_*|^2 \mu_t(dy, dv_*) \right)^{\frac{1}{2}} \]
\[ \leq (m_0 m_2)^{\frac{1}{2}}. \]

For the Lipschitz continuity of $a$, we use $|\psi'| \leq \beta$ and the mean value theorem
\[ |\psi(|x - z| - \psi(|y - z|)| \leq \beta |x - y| \]
to find
\[ |a(x, \mu_t) - a(y, \mu_t)| \leq \int_{\mathbb{R}^d} |\psi(|x - z| - \psi(|y - z|)| |v_*| \mu_t(dz, dv_*) \]
\[ \leq \beta |x - y|(m_0 m_2)^{\frac{1}{2}}. \]

(ii) By direct estimates, we have
\[ |b(x, \mu_t)| \leq \int_{\mathbb{R}^d} \psi(|x - y|) \mu_t(dy, dv_*) \leq m_0, \]
\[ |b(x, \mu_t) - b(y, \mu_t)| \leq \int_{\mathbb{R}^d} |\psi(|x - \bar{y}| - \psi(|y - \bar{y}|)| \mu_t(dy, dv_*) \leq \beta |x - y|. \]

(iii) We use the estimates (i) and (ii) to get
\[ |F(x, v, \mu_t) - F(y, v_*, \mu_t)| \leq |a(x, \mu_t) - a(y, \mu_t)| + |b(x, \mu_t) v - b(y, \mu_t) v_*| \]
\[ \leq |a(x, \mu_t) - a(y, \mu_t)| + |b(x, \mu_t) - b(y, \mu_t)| |v| + |b(y, \mu_t)| |v - v_*| \]
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\[ \leq \beta |x - y|(m_0 m_2)^{\frac{1}{2}} + \beta |x - y| P(t) + m_0 |v - v_*| \]
\[ = \beta \left( (m_0 m_2)^{\frac{1}{2}} + P(t) \right) |x - y| + m_0 |v - v_*| \]
\[ \leq \beta \left( (m_0 m_2)^{\frac{1}{2}} + P(t) \right) (|x, v| - (y, v_*)). \]

Here we used \( \max\{|x - y|, |v - v_*|\} \leq |x, v| - (y, v_*)|. \]

For \((x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T)\) and \(\mu \in L^\infty([0, T); \mathcal{M}(\mathbb{R}^d))\), we denote by \((X_\mu(s; t, x, v), V_\mu(s; t, x, v))\) to be the particle trajectory passing through \((x, v)\) at time \(t\), i.e.,

\[
\frac{d}{ds} X_\mu(s; t, x, v) = V_\mu(s; t, x, v), \quad s \in [0, T), \tag{5.6}
\]

\[
\frac{d}{ds} V_\mu(s; t, x, v) = -\lambda \int_{\mathbb{R}^d} \psi(|X_\mu(s; t, x, v) - y|)(V_\mu(s; t, x, v) - v_*)\mu_s(dy, dv_*), \tag{5.7}
\]

subject to initial data

\[ X_\mu(t; t, x, v) = x, \quad V_\mu(t; t, x, v) = v. \]

For notational simplicity, we use a simplified notation:

\[ [x_\mu(s), v_\mu(s)] := [x_\mu(s; t, x, v), v_\mu(s; t, x, v)]. \]

**Lemma 5.4.** Let \(\mu \in L^\infty([0, T); \mathcal{M}(\mathbb{R}^d))\) be a measure valued function with the properties:

1. A compact support for each time slice: For some nonnegative locally bounded functions \(R(t), P(t)\),

\[ \text{sp}(\mu(t)) \subset B_{R(t)}(0) \times B_{P(t)}(0). \]

2. Uniform boundedness of the first two moments:

\[ \int_{\mathbb{R}^d} \mu_1(dx, dv) \leq m_0 < \infty, \quad \int_{\mathbb{R}^d} |v|^2 \mu_1(dx, dv) \leq m_2 < \infty. \]

Then we have

(i) For any fixed \((x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [0, T)\), there is the unique global particle trajectories \([x_\mu(s; t, x, v), v_\mu(s; t, x, v)]\) which is a \(C^1\)-function of \(s \in [0, T]\) and admits an unique inverse map in the form of

\[ x := X_\mu(t; s, x, \bar{v}), \quad v := V_\mu(t; s, \bar{x}, \bar{v}). \]

where

\[ \bar{x} := X_\mu(s; t, x, v), \quad \bar{v} := V_\mu(s; t, x, v). \]

(ii) Let \(\hat{P}(t)\) and \(\hat{R}(t)\) be the bounds for supports of the velocity and position variables respectively. Then we have:

\[ \hat{P}(t) \leq \hat{P}(0) + \lambda (m_0 m_2)^{\frac{1}{2}} t, \quad \hat{R}(t) \leq \hat{R}(0) + \hat{P}(0) t + \frac{\lambda}{2} (m_0 m_2)^{\frac{1}{2}} t^2. \]

**Proof.** (i) Consider the vector field on the phase-space generated by \((v, \mathbf{A}\mathbf{F}(x, v, \mu_t))\). Then this vector field satisfies the following two properties:
1. Continuity in time $t$ due to the weak continuity of $\mu_t$.

2. Global Lipschitz continuity in $x$ and $v$ variables due to (iii) of Lemma 5.3.

Hence the standard ordinary differential equation theory implies that there exists the unique global $C^1$ particle trajectory passing through $(x,v)$ at time $t$ generated by the above vector field. And its inverse map is just a backward trajectory. Hence (i) holds.

(ii) Since the estimate for $\hat{R}(t)$ follows from the estimate for $\hat{P}(t)$, we only consider the growth estimate of $\hat{P}(t)$. For simplicity in presentation, we denote

$$[x(s), v(s)] := [x_{\mu}(s; 0, x, v), v_{\mu}(s; 0, x, v)],$$

with $(x, v) \in B_{\hat{R}(0)}(0) \times B_{\hat{P}(0)}(0)$.

We now consider the equation for $v$:

$$\frac{dv(s)}{ds} = \lambda a(x(s), \mu_s) - \lambda b(x(s), \mu_s)v(s).$$

We integrate the equation to get the integral equation:

$$v(t) = v_0 e^{-\lambda \int_0^t b(x(s), \mu_s)ds} + \lambda \int_0^t e^{-\lambda \int_s^t b(x(t), \mu_t)dt} a(x(s), \mu_s)ds.$$

Then we use $b \geq 0$ and the estimates in Lemma 5.3 to get

$$|v(t)| \leq |v_0| + \lambda \int_0^t |a(x(s), \mu_s)|ds \leq \hat{P}(0) + \lambda (m_0 m_2)^{1/2} t.$$

Hence we obtain the upper bound on the size of the velocity support of $\mu$:

$$\hat{P}(t) \leq \hat{P}(0) + \lambda (m_0 m_2)^{1/2} t.$$

Now we can directly get a bound on $x(t)$:

$$|x(t)| \leq |x(0)| + \int_0^t |v(s)|ds \leq \hat{R}(0) + \hat{P}(0) + \frac{\lambda}{2} (m_0 m_2)^{1/2} t^2.$$

This completes the proof of Lemma. $\square$

**Lemma 5.5.** For any $T \in (0, \infty]$, let $\mu \in L^\infty([0, T]; M(\mathbb{R}^{2d}))$ be a measure-valued solution of (5.2) satisfying (5.3)-(5.4). Then for any test function $h \in C^1_0(\mathbb{R}^{2d})$, we have

$$\int_{\mathbb{R}^{2d}} h(x, v) \mu_t(dx, dv) = \int_{\mathbb{R}^{2d}} h(X_{\mu}(t; s, x, v), V_{\mu}(t; s, x, v)) \mu_s(dx, dv).$$

Here $[X_{\mu}(t; s, x, v), V_{\mu}(t; s, x, v)]$ is the particle trajectory passing through $(x, v)$ at time $s$.

**Proof.** Recall that the defining relation (5.3) can be rewritten as

$$\int_{\mathbb{R}^{2d}} g(x, v, t) \mu_t(dx, dv) - \int_{\mathbb{R}^{2d}} g(y, v, s) \mu_s(dy, dv_s)$$

$$= \int_s^t \int_{\mathbb{R}^{2d}} (\partial_x g + v \cdot \nabla g + \lambda F(y, v) \cdot \nabla_y g) \mu_s(dy, dv)d\tau. \quad (5.8)$$
We now choose a test function $g$ so that the R.H.S. of (5.8) vanishes. For any $h \in C^1_0(\mathbb{R}^{2d})$ and fixed $t$, we set
\[
g(\bar{x}, \bar{v}, \tau) := h(X_\mu(t; \tau, \bar{x}, \bar{v}), V_\mu(t; \tau, \bar{x}, \bar{v})).
\]
Then, by (i) in Lemma 5.4, we have
\[
g(X_\mu(\tau; t, \bar{x}, \bar{v}), V_\mu(\tau; t, \bar{x}, \bar{v}), \tau) = h(\bar{x}, \bar{v}). \tag{5.9}
\]
Direct differentiation of the above relation (5.9) with respect to $\tau$ and Lemma 5.4 imply
\[
g \in C^1_0(\mathbb{R}^{2d} \times [0, T]), \quad \partial_\tau g + \bar{v} \cdot \nabla g + \lambda F(\bar{y}, \bar{v}) \cdot \nabla \bar{v} g = 0.
\]
Hence the relation (5.8) implies
\[
\int_{\mathbb{R}^{2d}} g(x, v, t) \mu_t(dx, dv) = \int_{\mathbb{R}^{2d}} g(x, v, s) \mu_s(dx, dv),
\]
or
\[
\int_{\mathbb{R}^{2d}} h(x, v) \mu_t(dx, dv) = \int_{\mathbb{R}^{2d}} h(X_\mu(t; s, \bar{x}, \bar{v}), V_\mu(t; s, \bar{x}, \bar{v}))) \mu_s(dx, dv).
\]

5.3. Stability estimate in bounded Lipschitz distance
In this part, we derive a stability estimate for measure valued solutions to (5.2) in a bounded Lipschitz distance. This stability estimate is crucial in the mean-field limit of the C-S particle model in next section.

We review the definition of the bounded Lipschitz distance in [13, 17]. We first define the admissible set $\Omega$ of test functions:
\[
\Omega := \left\{ g : \mathbb{R}^{2d} \to \mathbb{R} : \|g\|_{L^\infty} \leq 1, \quad \text{Lip}(g) := \sup_{z_1 \neq z_2 \in \mathbb{R}^{2d}} \frac{|g(z_1) - g(z_2)|}{|z_1 - z_2|} \leq 1 \right\}.
\]

**Definition 5.6.** [13, 17] Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^{2d})$ be two Radon measures. Then the bounded Lipschitz distance $d(\mu, \nu)$ between $\mu$ and $\nu$ is given by
\[
d(\mu, \nu) := \sup_{g \in \Omega} \left| \int_{\mathbb{R}^{2d}} g(x, v) \mu(dx, dv) - \int_{\mathbb{R}^{2d}} g(x, v) \nu(dx, dv) \right|.
\]

**Remark 5.2.**
1. The space of Radon measures $\mathcal{M}(\mathbb{R}^{2d})$ equipped with the metric $d(\cdot, \cdot)$ is a complete metric space.
2. The bounded Lipschitz distance $d$ is equivalent to the Wasserstein-1 distance (Kantorovich-Rubinstein distance) $W_1$ (see [7, 17]):
\[
W_1(\mu, \nu) := \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^{2d}} |z_1 - z_2| \gamma(dz_1, dz_2),
\]
where $\Pi(\mu, \nu)$ is the set of all product measures on $\mathbb{R}^d \times \mathbb{R}^d$ such that their marginals
are $\mu$ and $\nu$.

3. For any $g \in C_0(\mathbb{R}^d)$ with $|g(z)| \leq a$ and $\text{Lip}(g) \leq b$, we have

$$\left| \int_{\mathbb{R}^d} g(x, v) \mu(dx, dv) - \int_{\mathbb{R}^d} g(x, v) \nu(dx, dv) \right| \leq \max\{a, b\} d(\mu, \nu). \quad (5.10)$$

We present a series of estimates regarding the dynamics of particle trajectories.

**Lemma 5.7.** Let $\mu, \nu \in L^\infty([0, T]; \mathcal{M}(\mathbb{R}^d))$ be two measure valued functions with comp-
act supports for each time slice:

$$\text{spt}(\mu_t), \text{spt}(\nu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0).$$

where $R(t)$ and $P(t)$ are nonnegative locally bounded functions. Then $a(x, \mu_t)$ and

$b(x, \mu_t)$ in (5.5) satisfy

(i) $|a(x, \mu_t) - a(x, \nu_t)| \leq d \max\{P(t), \beta P(t) + 1\} d(\mu_t, \nu_t)$,

(ii) $|b(x, \mu_t) - b(x, \nu_t)| \leq \max\{1, \beta\} d(\mu_t, \nu_t)$.

**Proof.** (i) Let $(y, v_*) \in \text{spt}(\mu_t) \cup \text{spt}(\nu_t)$. Then we have

$$|\psi(|x - y|)v_*^i| \leq P(t) \quad \text{and}$$

$$|\psi(|x - y_1|)v_*^i - \psi(|x - y_2|)v_*^j| \leq (\beta P(t) + 1)(|y_1, v_*| - |y_2, v_*|).$$

It follows from (5.10) that we have

$$|a^i(x, \mu_t) - a^i(x, \nu_t)| = \int_{\mathbb{R}^d} |\psi(|x - y|)v_*^i(\mu_t(dy, dv_*), \nu_t(dy, dv_*))|$$

$$\leq \max\{P(t), \beta P(t) + 1\} d(\mu_t, \nu_t).$$

This yields the desired result.

(ii) Note that

$$\psi(|x - y|) \leq 1 \quad \text{and} \quad |\psi(|x - y_1|) - \psi(|x - y_2|)| \leq \beta|y_1 - y_2|,$$

hence we again use (5.10) to get

$$|b(x, \mu_t) - b(x, \nu_t)| \leq \int_{\mathbb{R}^d} |\psi(|x - y|)(\mu_t(dy, dv_*), \nu_t(dy, dv_*))|$$

$$\leq \max\{1, \beta\} d(\mu_t, \nu_t).$$

**Lemma 5.8.** Let $\mu, \nu \in L^\infty([0, T]; \mathcal{M}(\mathbb{R}^d))$ be two measure valued functions with comp-
act supports

$$\text{spt}(\mu_t), \text{spt}(\nu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0),$$

and finite moments

$$\int_{\mathbb{R}^d} \mu_t(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^d} |v|^2 \mu_t(dx, dv) \leq m_2,$$
Here we used the simplified notations:

\[ X_\mu(s) := X_\mu(s; t, x, v), \quad X_\nu(s) := X_\nu(s; t, x, v). \]

**Proof.** (i) By definition of \( a \), we have

\[
|a(X_\mu(s), \mu_s) - a(X_\nu(s), \nu_s)| \\
= \left| \int_{\mathbb{R}^2^d} \psi(|X_\mu - y|) v_s \mu_s(dy, dv_s) - \int_{\mathbb{R}^2^d} \psi(|X_\nu - y|) v_s \nu_s(dy, dv_s) \right| \\
\leq \left| \int_{\mathbb{R}^2^d} \left( \psi(|X_\mu - y|) - \psi(|X_\nu - y|) \right) v_s \mu_s(dy, dv_s) \right| \\
+ \left| \int_{\mathbb{R}^2^d} \psi(|X_\nu - y|) v_s \nu_s(dy, dv_s) \right| \\
: = \mathcal{I}_1 + \mathcal{I}_2.
\]

The term \( \mathcal{I}_1 \) can be treated as follows.

\[
\mathcal{I}_1 \leq \beta |X_\mu - X_\nu| \int_{\mathbb{R}^2^d} |v_s| \mu_s(dy, dv_s) \leq \beta |X_\mu - X_\nu|(m_0 m_2)^{1\over 2}.
\]

On the other hand, for the estimate of \( \mathcal{I}_2 \), we use the same argument as in Lemma 5.7 (i) to get

\[
\mathcal{I}_2 \leq d \max\{P(\max\{s, t\}), \beta P(\max\{s, t\}) + 1\} d(\mu_s, \nu_s).
\]

(ii) Similar to (i), we have

\[
|b(X_\mu(s), \mu_s) - b(X_\nu(s), \nu_s)| \\
\leq \int_{\mathbb{R}^2^d} \left| \psi(|X_\mu - y|) - \psi(|X_\nu - y|) \right| \mu_s(dy, dv_s) \\
+ \int_{\mathbb{R}^2^d} \psi(|X_\nu - y|) \mu_s(dy, dv_s) - \int_{\mathbb{R}^2^d} \psi(|X_\nu - y|) \nu_s(dy, dv_s) \\
\leq \beta |X_\mu - X_\nu| m_0 + \max\{1, \beta\} d(\mu_s, \nu_s).
\]

\[ \square \]

**Lemma 5.9.** Let \( \mu, \nu \in L^\infty([0, T]; \mathcal{M}(\mathbb{R}^2^d)) \) be two measure valued solutions with compact supports

\[ \text{spt}(\mu_1), \text{spt}(\nu_1) \subset B_{R(t)}(0) \times B_{P(t)}(0), \quad \text{for some positive functions } R(t), P(t), \]

and uniform bounded moments

\[ \int_{\mathbb{R}^2^d} \mu_t(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^2^d} |v|^2 \mu_t(dx, dv) \leq m_2, \]

Then for any \( 0 \leq s \leq T \), we have

(i) \[ |a(X_\mu(s), \mu_s) - a(X_\nu(s), \nu_s)| \leq \beta(m_0 m_2)^{1\over 2} |(X_\mu(s) - X_\nu(s))| + d \max\{P(\max\{s, t\}), \beta P(\max\{s, t\}) + 1\} d(\mu_s, \nu_s). \]

(ii) \[ |b(X_\mu(s), \mu_s) - b(X_\nu(s), \nu_s)| \leq \beta m_0 |(X_\mu(s) - X_\nu(s))| + \max\{1, \beta\} d(\mu_s, \nu_s). \]

Here we used the same argument as in Lemma 5.7 (i) to get

\[ \mathcal{I}_2 \leq d \max\{P(\max\{s, t\}), \beta P(\max\{s, t\}) + 1\} d(\mu_s, \nu_s). \]
\[
\int_{\mathbb{R}^d} \nu_t(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^d} |v|^2 \nu_t(dx, dv) \leq m_2.
\]

Then for any \(0 \leq s \leq T\), we have
\[
|X_{\mu}(s; t, v) - X_{\nu}(s; t, v)| + |V_{\mu}(s; t, v) - V_{\nu}(s; t, v)|
\leq \int_{\min\{s, t\}}^{\max\{s, t\}} \alpha(t; s) d(\mu_t, \nu_t) d\tau,
\]

where \(\alpha = \alpha(t; s)\) is a smooth function depends only on \(T, \lambda, \beta, d, m_0, m_2, P(\cdot)\) and \(R(\cdot)\).

**Proof.** We set
\[
y(s) := X_{\mu}(s; t, v) - X_{\nu}(s; t, v), \quad v(s) := V_{\mu}(s; t, v) - V_{\nu}(s; t, v).
\]

We consider the case \(0 < t < s\). The other case is exactly the same except the change of sign. It follows from (5.6) that we have
\[
\frac{dy(\tau)}{d\tau} = v(\tau), \quad y(t) = 0.
\]

We integrate the above equation from \(\tau = t\) to \(\tau = s\) to find
\[
|y(s)| \leq \int_t^s |v(\tau)| d\tau.
\]

On the other hand, \(v(\tau)\) satisfies
\[
\begin{align*}
\frac{dv(\tau)}{d\tau} &+ \lambda b(X_{\mu}(\tau; t, v), \mu_\tau) v(\tau) \\
&= \lambda |a(X_{\mu}(\tau; t, v), \mu_\tau) - a(X_{\nu}(\tau; t, v), \nu_\tau)| \\
&- \lambda |b(X_{\mu}(\tau; t, v), \mu_\tau) - b(X_{\nu}(\tau; t, v), \nu_\tau)| V_{\nu}(\tau; t, v), \\
v(t) &= 0.
\end{align*}
\]

We use the method of integrating factor and \(e^{-\lambda b(X_{\mu})} \leq 1\) to get
\[
|v(s)| \leq \int_t^s \lambda |a(X_{\mu}(\tau; t, v), \mu_\tau) - a(X_{\nu}(\tau; t, v), \nu_\tau)| d\tau \\
+ \int_t^s \lambda |b(X_{\mu}(\tau; t, v), \mu_\tau) - b(X_{\nu}(\tau; t, v), \nu_\tau)| |V_{\nu}(\tau; t, v)| d\tau \\
\leq \int_t^s \lambda \left[ d\max\{P(\tau), \beta P(\tau) + 1\} + \max\{1, \beta\} P(\tau) \right] d(\mu_\tau, \nu_\tau) d\tau \\
+ \int_t^s \lambda \beta \left[ m_0 P(\tau) + (m_0 m_2)^{1/2} \right] |y(\tau)| d\tau \\
\leq \int_t^s \lambda \left[ d + (1 + d) P(\tau) \max\{1, \beta\} \right] d(\mu_\tau, \nu_\tau) d\tau \\
+ \int_t^s \lambda \beta \left[ m_0 P(\tau) + (m_0 m_2)^{1/2} \right] |y(\tau)| d\tau.
\]

Here we used the fact that
\[
d\max\{P(\tau), \beta P(\tau) + 1\} + \max\{1, \beta\} P(\tau) \leq d + (1 + d) P(\tau) \max\{1, \beta\}.
\]
Then \( z(s) = |v(s)| + |y(s)| \), \( A(\tau) := \lambda [d + (1 + d)P(\tau) \max \{1, \beta\}] \),
\( B(\tau) := \max \left\{ 1, \lambda \left[ m_0 P(\tau) + (m_0 m_2)^{1/2} \right] \right\} \).

Then \( z \) satisfies the Gronwall type inequality:
\[
z(s) \leq \int_t^s A(\tau) d(\mu_\tau, \nu_\tau) \, d\tau + \int_t^s B(\tau) z(\tau) \, d\tau
\]

Hence the Gronwall’s Lemma implies
\[
z(s) \leq \int_t^s A(\tau) e^{\int_t^\tau B(\tau) \, d\tau} d(\mu_\tau, \nu_\tau) \, d\tau \equiv \int_t^s \alpha(\tau; s) d(\mu_\tau, \nu_\tau) \, d\tau.
\]

Here we used a simplified notation:
\[
\alpha(\tau; s) := A(\tau) e^{\int_t^\tau B(\tau) \, d\tau}.
\]

Therefore, we have
\[
|X_\mu(s; t, x, v) - X_\nu(s; t, x, v)| + |V_\mu(s; t, x, v) - V_\nu(s; t, x, v)|
\]
\[
= |v(s)| + |y(s)| \leq z(s) \leq \int_t^s \alpha(\tau; s) d(\mu_\tau, \nu_\tau) \, d\tau.
\]

\[\square\]

**Proposition 5.10.** Let \( \mu, \nu \in L^\infty([0, T]; \mathcal{M}(\mathbb{R}^d)) \) be two measure valued solutions corresponding initial data \( \mu_0, \nu_0 \) with compact supports:

\[\text{spt}(\mu_t), \text{spt}(\nu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0), \text{ for some positive functions } R(t), P(t),\]

and uniform bounded moments:

\[
\int_{\mathbb{R}^d} \mu_0(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^d} |v|^2 \mu_0(dx, dv) \leq m_2,
\]

\[
\int_{\mathbb{R}^d} \nu_0(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^d} |v|^2 \nu_0(dx, dv) \leq m_2.
\]

Then there exists a nonnegative function \( C_2 = C(T, d, \lambda, \beta, P(\cdot), R(\cdot), m_0, m_2) \) satisfying
\[
d(\mu_t, \nu_t) \leq C_2 d(\mu_0, \nu_0), \quad t \in [0, T).
\]

**Proof.** Let \( g \in C_0(\mathbb{R}^d) \) a test function in \( \Omega \), i.e.,
\[
\|g\|_{L^\infty} \leq 1 \quad \text{and} \quad \text{Lip}(g) \leq 1.
\]

Then we have
\[
\left| \int_{\mathbb{R}^d} g(x, v) \mu_t(dx, dv) - \int_{\mathbb{R}^d} g(x, v) \nu_t(dx, dv) \right|
\]
\[
\leq \int_{\mathbb{R}^d} |g(X_\mu(t; 0, x, v), V_\mu(t; 0, x, v)) - g(X_\nu(t; 0, x, v), V_\nu(t; 0, x, v))| \mu_0(dx, dv)
\]
Lemma 6.1. Let $\mu$ a finite sum. Note that since $\mu$, hence the whole phase space $R^d$, the relative velocity factor $(v - v_s)$ in Sphohn’s case in [13, 17], the kinetic model (5.2) do not have a bounded kernel due to a bounded and Lipshitz kernel in [13, 17]. However unlike to the case in Neuzert and Radon measure $\mu$, this section, we show the existence of measure valued solutions to (5.2) with initial $\mu$. We construct the initial approximation $\mu_0$, suppose initial Radon measure $\mu$ has a compact support, i.e.,

$\exists \ R_0, P_0 < \infty$ such that $\text{spt}(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0)$.

We construct the initial approximation $\mu_0^h$ as

$$\mu_0^h := \sum_{i,j} m_{ij} \delta(x - x_i) \otimes \delta(v - v_j), \quad m_{ij} := \int_{R^d(i,j)} \mu_0(dx, dv). \tag{6.1}$$

Note that since $\mu_0$ has a compact support, the sum in the above definition (6.1) is in fact a finite sum.

Lemma 6.1. Let $\mu_0$ be a given initial Radon measure on $R^{2d}$ with a compact support:

$\text{spt}(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0)$.
and let $\mu_0^h$ be the initial approximation given by (6.1) with a uniform grid size $h$. Then we have

$$d(\mu_0^h, \mu_0) \leq \sqrt{\frac{d}{2}} \|\mu_0\| h.$$  

Here $d(\cdot, \cdot)$ is the bounded Lipschitz distance introduced in Section 5.3, and $\|\mu_0\| := \int_{\mathbb{R}^d} \mu_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v}$.

**Proof.** Let $g \in \Omega$ be a test function with the properties ($|g| \leq 1$, $\text{Lip}(g) \leq 1$). Then we have

$$\left| \int_{\mathbb{R}^d} g(\mathbf{x}, \mathbf{v}) \mu_0(\mathbf{x}, d\mathbf{v}) - \int_{\mathbb{R}^d} g(\mathbf{x}, \mathbf{v}) \mu_0^h(\mathbf{x}, d\mathbf{v}) \right|$$

$$\leq \sum_{i,j} \left| \int_{R^h(i,j)} \mu_0(\mathbf{x}, d\mathbf{v}) - \int_{R^h(i,j)} \mu_0^h(\mathbf{x}, d\mathbf{v}) \right|$$

$$\leq \sum_{i,j} \int_{R^h(i,j)} |g(\mathbf{x}, \mathbf{v}) - g(\mathbf{x}, \mathbf{v})| \mu_0(\mathbf{x}, d\mathbf{v})$$

$$\leq \sum_{i,j} \int_{R^h(i,j)} |(\mathbf{x}, \mathbf{v}) - (\mathbf{x}, \mathbf{v})| \mu_0(\mathbf{x}, d\mathbf{v})$$

$$\leq \sqrt{\frac{d}{2}} \|\mu_0\| h.$$ 

\[\square\]

**Remark 6.1.** 1. We will use single index $i, i = 1, \cdots, N$ instead of double index, and rewrite $\mu_0^h$ in (6.1) as

$$\mu_0^h := \sum_{i=1}^N m_i \delta(\mathbf{x} - \mathbf{x}_i) \otimes \delta(\mathbf{v} - \mathbf{v}_i).$$

2. Note that the sequence $\{\mu_0^h\}$ satisfies

$$d(\mu_0^h, \mu_0^h) \leq d(\mu_0^h, \mu_0) + d(\mu_0, \mu_0^h) \leq \sqrt{d} \|\mu_0\| \max\{h_1, h_2\},$$

which implies that $\{\mu_0^h\}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}(\mathbb{R}^d), d(\cdot, \cdot))$.

The particle method is exactly identical with the Cucker-Smale’s dynamics:

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \lambda \sum_{j=1}^N m_j \psi(|x_i - x_j|)(v_j - v_i), \quad i = 1, \cdots, N. \quad (6.2)$$

With solutions $(\mathbf{x}_i(t), \mathbf{v}_i(t))$ of (6.2), we can define an approximate solution:

$$\mu_t^h := \sum_{i=1}^N m_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \otimes \delta(\mathbf{v} - \mathbf{v}_i(t)). \quad (6.3)$$

It is clearly that bounds for supports of the velocity and position variables as $\tilde{P}^h(t)$ and $\tilde{R}^h(t)$ is also bounds of support of measure (6.3) $\mu_t^h$, i.e.,

$$\text{spt}(\mu_t^h) \subset B_{\tilde{P}^h(t)}(0) \times B_{\tilde{R}^h(t)}(0). \quad (6.4)$$
6.2. Convergence of a particle method

In this part, we present a convergence of approximate solutions to the measure valued solution of (5.2) and error estimate for the particle method which is first order accurate.

**Theorem 6.2.** Let \( \mu_0 \in M(\mathbb{R}^{2d}) \) be a Radon measure with a compact support. Let \( \mu^h_t \) be the approximate solution constructed by the particle method with a grid size \( h \). Let \( (R_0, P_0) \) be the radius of the initial compact support, i.e.,

\[
\text{spt}(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0),
\]

and let \( m_0,m_2 \) be the moments of the initial measure

\[
m_0 = \int_{\mathbb{R}^{2d}} \mu_0(dx, dv), \quad m_2 = \int_{\mathbb{R}^{2d}} |v|^2 \mu_0(dx, dv).
\]

Then there is a measure valued solution \( \mu \in L^\infty([0,T); M(\mathbb{R}^{2d})) \) to (5.2) with initial data \( \mu_0 \) such that

1. \( \mu \) is the weak-* limit of the approximate solutions \( \mu^h_t \) as \( h \to 0^+ \), and we have the error estimate

\[
d(\mu_t, \mu^h_t) \leq C_3 h,
\]

where \( C_3 = C_3(T,d,\lambda,\beta,P_0,m_0,m_2) \) is a positive constant depending on the arguments specified.

2. \( \mu \) is weakly Lip continuous and has bounded first two moments for each time slice:

\[
\int_{\mathbb{R}^{2d}} \mu_t(dx, dv) \leq m_0, \quad \int_{\mathbb{R}^{2d}} |v|^2 \mu_t(dx, dv) \leq m_2.
\]

Moreover \( \mu \) has compact support for each time slice:

\[
\text{spt}(\mu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0),
\]

where \( R(t) \) and \( P(t) \) are positive bounded functions satisfying growth estimates:

\[
P(t) \leq P_0 + \lambda (m_0m_2)^{\frac{1}{2}} t, \quad R(t) \leq R_0 + P_0 t + \frac{\lambda}{2} (m_0m_2)^{\frac{1}{2}} t^2
\]

3. \( \mu \) is unique in the class of measure valued solution to (5.2) with initial data \( \mu_0 \) and compact supports for each time slice.

**Proof.** We present its proof in several steps.

**Step A (bounds for moments and compact support):** Denote

\[
m_0^h(t) = \int_{\mathbb{R}^{2d}} \mu^h_t(dx, dv), \quad m_2^h(t) = \int_{\mathbb{R}^{2d}} |v|^2 \mu^h_t(dx, dv),
\]

and let \( (R^h(t), P^h(t)) \) be the radius of the initial compact support, i.e.,

\[
\text{spt}(\mu^h_t) \subset B_{R^h(t)}(0) \times B_{P^h(t)}(0).
\]

As explained in (6.4) that \( P^h(t) \) and \( R^h(t) \) can also be bounds for supports of the velocity and position variables \( \hat{P}^h(t) \) and \( \hat{R}^h(t) \) respectively. From Remark 5.1 (3),
Since a complete metric space we know that 

\[ m_0^h(t) = m_0^h(0), \quad m_2^h(t) \leq m_2^h(0), \]

\[ P^h(t) \leq P^h(0) + \lambda(m_0^h(0)m_2^h(0))^{\frac{1}{2}}t, \quad R^h(t) \leq R^h(0) + P^h(0)t + \frac{\lambda}{2}(m_0^h m_2^h)^{\frac{1}{2}}t^2 \]

Due to the construction of initial approximation \( \mu_0^h \), it is easy to prove that

\[ m_0^h(0) \leq m_0 + C, \quad m_2^h(0) \leq m_2 + C, \quad R^h(0) \leq R_0 + C, \quad P^h(0) \leq P_0 + C. \]  

(6.5)

Hence

\[ P^h(t) \leq P_0 + \lambda(m_0 m_2)^{\frac{1}{2}}t + C(1 + t), \]

(6.6)

\[ R^h(t) \leq R_0 + P_0 t + \frac{\lambda}{2}(m_0 m_2)^{\frac{1}{2}}t^2 + C(1 + t + t^2). \]  

(6.7)

Step B (Existence of weak-* limit and error estimate): For fixed \( t \in [0, T) \), it follows from Proposition 5.10 and Remark 6.1 that

\[ d(\mu_t^{h_1}, \mu_t^{h_2}) \leq C_3 d(\mu_0^{h_1}, \mu_0^{h_2}) \leq C_2 \sqrt{d} ||\mu_0|| \text{max}\{h_1, h_2\} \]

Due to (6.6) and (6.6) and fact that \( m_0 = ||\mu_0|| \), we know that \( C_2 \) depends only on \( T, d, \lambda, \beta, P(0), m_0 \) and \( m_2 \). Hence, we set

\[ C_3(T, d, \lambda, \beta, P_0, m_0, m_2) := C_2 \sqrt{d} ||\mu_0|| \]

to get

\[ d(\mu_t^{h_1}, \mu_t^{h_2}) \leq C_3 \text{max}\{h_1, h_2\} \]

Therefore the sequence of approximate solutions \( \{\mu_t^h\} \) is a Cauchy sequence in the complete metric space \( (\mathcal{M}^2, d(\cdot, \cdot)) \). This guarantees the existence of a limit measure \( \mu_t \in \mathcal{M}^2 \). By taking \( h_1 = h \) fixed and and \( h_2 \rightarrow 0 \), we have a first order accurate error estimate

\[ d(\mu_t^h, \mu_t) \leq C_3 h. \]

Since \( d(\cdot, \cdot) \)-convergence is also equivalent to the weak-* convergence[17], we know that \( \mu_t \) is the weak-* limit of \( \mu_t^h \). From (6.5)- (6.7), we have

\[ m_0(t) = m_0, \quad m_2(t) \leq m_2, \]

\[ P(t) \leq P_0 + \lambda(m_0 m_2)^{\frac{1}{2}}t, \quad R(t) \leq R_0 + P_0 t + \frac{\lambda}{2}(m_0 m_2)^{\frac{1}{2}}t^2. \]

Step C (Weakly Lipschitz continuity): We need to check the weakly continuity of the map \( t \rightarrow \mu_t \). We first note that the vector field \( (v, \lambda F(\mu_t^h)) \) is bounded in the time-strip \( \mathbb{R}^{2d} \times [0, T] \):

\[ |(v, \lambda F(\mu_t^h))| \leq |v| + \lambda |F(\mu_t^h)| \leq |v| + \lambda |a(x, \mu_t^h)| + \lambda |b(x, \mu_t^h)| |v|. \]
Here we used Lemma 5.3 and 5.4. This leads
\[|X^h(t + \Delta t) - X^h(t)| + |V^h(t + \Delta t) - V^h(t)| \leq C(T, d, \lambda, \beta, P_0, m_0, m_2)\Delta t.\]

On the other hand, note that for any \( g \in C_0^1(\mathbb{R}^{2d}) \), we have from Lemma 5.5 that
\[
\begin{align*}
|\langle \mu^h_{t+\Delta t}, g \rangle - \langle \mu^h_t, g \rangle | & = \left| \int_{\mathbb{R}^{2d}} (g(X^h(t + \Delta t; t, x, v), V^h(t + \Delta t; t, x, v)) - g(x, v)) \mu^h_t(\mathrm{d}x, \mathrm{d}v) \right| \\
& \leq \|g\|_{C_1} m_0 \left( |X^h(t + \Delta t) - X^h(t)| + |V^h(t + \Delta t) - V^h(t)| \right) \\
& \leq C(T, d, \lambda, \beta, P_0, m_0, m_2, g)\Delta t.
\end{align*}
\]

As \( \Delta t \to 0 \), we see that \( \mu \) is weakly Lipschitz continuous in \( t \). Since \( \mu^h \) satisfy required bounds, the limit \( \mu \) also satisfy required bounds.

Step D (\( \mu \) satisfies (5.3)): In the following, we show that this limit measure \( \mu \) is a measure valued solution to (5.2).

Note that \( \mu^h \) is a measure valued solution, hence it satisfies the equation (5.2):
\[
\langle \mu^h_t, g(\cdot, t) \rangle - \langle \mu^h_0, g(\cdot, 0) \rangle = \int_0^t \langle \mu^h_s, \partial_s g + v \cdot \nabla_x g + \lambda F(\mu^h_s) \cdot \nabla v g \rangle \mathrm{d}s.
\]

Since \( d(\cdot, \cdot) \)-convergence is equivalent to weak-* convergence, it is easy to see that the first two terms in (6.8) converges to the corresponding terms for weak-* limit \( \mu_t \), i.e.,
\[
\langle \mu^h_t, g(\cdot, t) \rangle - \langle \mu^h_0, g(\cdot, 0) \rangle \to \langle \mu_t, g(\cdot, t) \rangle - \langle \mu_0, g(\cdot, 0) \rangle \text{ as } h \to 0+.
\]

It remains to show that the terms in the R.H.S. of the equation (6.8) converges to the corresponding terms for \( \mu \).

We claim: For any test function \( g \in C_0^1(\mathbb{R}^{2d} \times [0, T)) \),
\[
\int_0^t \langle \mu^h_s, \partial_s g + v \cdot \nabla_x g + \lambda F(\mu^h_s) \cdot \nabla v g \rangle \mathrm{d}s \to \int_0^t \langle \mu_s, \partial_s g + v \cdot \nabla_x g + \lambda F(\mu_s) \cdot \nabla v g \rangle \mathrm{d}s,
\]
as \( h \to 0+ \).

The proof of claim: The above relation directly follows from the following strong result. For \( t \in [0, T) \), we prove that
\[
\left| \langle \mu^h_t, \partial_s g + v \cdot \nabla_x g + \lambda F(\mu^h_s) \cdot \nabla v g \rangle - \langle \mu_t, \partial_s g + v \cdot \nabla_x g + \lambda F(\mu_t) \cdot \nabla v g \rangle \right| \\
\leq C_4 h.
\]

Here \( C_4 = C(T, d, \lambda, \beta, P_0, m_0, m_2, g) \) is a positive constant.

Note that
\[
\left| \langle \mu^h_t, \partial_s g + v \cdot \nabla_x g \rangle - \langle \mu_t, \partial_s g + v \cdot \nabla_x g \rangle \right| \leq \left| \partial_s g + v \cdot \nabla_x g \right|_{C_1(d(\mu^h_t, \mu_t)) \leq Ch},
\]
and hence it is enough to show that
\[
\left| \langle \mu^h_t, \lambda F(\mu^h_s) \cdot \nabla v g \rangle - \langle \mu_t, \lambda F(\mu_s) \cdot \nabla v g \rangle \right| \leq Ch.
\]
By definition of $F$, we have

$$\left| \langle \mu^*_b, \lambda F(\mu^*_b) \cdot \nabla v \rangle - \langle \mu_s, \lambda F(\mu_s) \cdot \nabla v \rangle \right|$$

$$\leq \lambda \left| \int_{\mathbb{R}^d} (F(\mu^*_b) - F(\mu_s)) \cdot \nabla v \mu^*_b \, dx \, dv \right|$$

$$+ \lambda \left| \int_{\mathbb{R}^d} F(\mu_s) \cdot \nabla v \mu^*_b \, dx \, dv \right| - \int_{\mathbb{R}^d} F(\mu_t) \cdot \nabla v \mu_s \, dx \, dv$$

$$:= J_1 + J_2.$$

We next estimate $J_i$, $i = 1, 2$ as follows.

Case 1 ($J_1$): We use Lemma 5.4 and Lemma 5.7 to see

$$|F(x, v, \mu_t) - F(x, v, \mu^*_b)| \leq |a(x, \mu_t) - a(x, \mu^*_b)| + |b(x, \mu_t) - b(x, \mu^*_b)||v|$$

$$\leq d \max\{P(t), \beta P(t) + 1\} d(\mu_t, \mu^*_b) + \max\{1, \beta\} P(t) d(\mu_t, \mu^*_b)$$

$$\leq C d(\mu_t, \mu^*_b)$$

$$\leq Ch.$$

This yields

$$J_1 \leq m_0 \lambda Ch.$$

Case 2 ($J_2$): In this case, we use Lemma 5.3 to get

$$||F(\mu_t)\cdot \nabla v||_{L^\infty} \leq ||F(\mu_t)|| \cdot ||\nabla v||_{L^\infty} \leq \left( m_0 m_2 \right)^{\frac{1}{2}} + m_0 P(t) \leq C,$$

$$|F(x, v, \mu_t) \cdot \nabla v| - F(y, v, \mu_t) \cdot \nabla v| \leq \left\{ \begin{array}{ll}
2 \beta \left( m_0 m_2 \right)^{\frac{1}{2}} + P(t) \right) + m_0 \right||\nabla v||_{L^\infty} + ||F||_{L^\infty} \cdot |g|_{C^2} \right\} (x, v) - (y, v_s)$$

$$\leq C ||(x, v) - (y, v_s)||.$$
REFERENCES