Existence of Weak Solutions to the Equations of Non-Stationary Motion of Non-Newtonian Fluids with Shear Rate Dependent Viscosity – J. Wolf

1. Introduction and statement of the result.

$$\partial_t u + \nabla \cdot (u \otimes u - S + pI) = -\nabla \cdot f, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

in $Q = \Omega \times (0, T)$,

$$\text{the initial condition } u(x,0) = u_0(x) \quad \text{for } |x \in \Omega, \tag{1.3}$$

$$\text{Dirichlet boundary condition}, \tag{1.4}$$

$f = \{f_{ij}\}$ external force

$D = (D_{ij}), \qquad D_{ij} = D_{ij}(u) := \frac{1}{2}(\partial_{x_j} u^i + \partial_{x_i} u^j)$

$D_{II} = \frac{1}{2} D : D = \frac{1}{2} D_{ij} D_{ij} =$ second invariant of D

$$S = \nu_0 (D_{II})^{\frac{q-2}{2}} D,$$

$$S = \nu_0 (1 + D_{II})^{\frac{q-2}{2}} D,$$

$$S = \nu_0 (D_{II})^{\frac{q-2}{2}} D + \nu_\infty D,$$

$$S = \nu_0 (D_{II})^{\frac{q-2}{2}} D + \nu_\infty D,$$

Basic Assumptions:

(I) $S : Q \times \mathbb{M}^n_{\text{sym}} \to \mathbb{M}^n_{\text{sym}}$ is a Carathéodory function;

(II) growth condition:

$\|S(x, t\xi)\| \leq c_0 \|\xi\|_{q-1} + g_1 \forall \xi \in \mathbb{M}^n_{\text{sym}}, a.e. (x,t) \in Q$, $(c_o > 0, g_1 \in L^{q'}(Q), g_1 \geq 0)$;

(III) coercivity:

$S(x, t\xi) : \xi \geq \nu_0 \|\xi\|_q - g_2 \forall \xi \in \mathbb{M}^n_{\text{sym}}, a.e. (x,t) \in Q$ $(\nu_0 > 0, g_2 \in L^1(Q), g_2 \geq 0)$;

(IV) strict monotonicity:

$\big(S(x,t,\xi) - S(x,t,\xi)\big) : (\xi - \eta) > 0 \forall \xi, \eta \in \mathbb{M}^n_{\text{sym}}, (\xi \neq \eta), a.e. (x,t) \in Q$

Let $(X, \|\cdot\|_X)$ be a normed space.
$L^q(o,T;X)$ the space of all Bochner measurabel functions $\phi : (0,T) \to X$, such that

$$\|\phi\|_{L^q(0,T;X)} := \Big(\int_0^T \|\phi(t)\|_X^q dt \Big)^{\frac{1}{q}} < \infty.$$

Let

$$\mathcal{V} \stackrel{\text{def}}{=} \{ v \in C_0^\infty(\Omega)^3 : \ \text{div } v = 0 \},$$

$$\mathbf{V}_q \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } W^{1,q}(\Omega)^3,$$

$$\mathbf{H} \stackrel{\text{def}}{=} \text{closure of } \mathcal{V} \text{ in } L^q(\Omega)^3,$$

where

$$\|\phi\|_{V_q} := \begin{cases} \|D(\phi)\|_{L^q(\Omega)} & \text{if } \Omega \text{ is bounded,} \\ \|D(\phi)\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} & \text{if } \Omega \text{ is unbounded.} \end{cases}$$

Definition 1.1. *Let* $\frac{2n}{n+2} \le q < \infty$. *Assume (I) and (II). Let* $f \in L^1(Q)$ *and* $u_0 \in H$. $u \in L^q(0,T;V_q) \cap L^\infty(0,T;H)$ *is called a weak solution to* (1.1)–(1.4)*: if*

$$-\int_Q u \cdot \partial_t \phi dxdt + \int_Q (S(x,t,D(u)) - u \otimes u) : D(\phi) dxdt = \int_Q f : \nabla \phi dxdt + \int_\Omega u_0 \cdot \phi(0) dx$$

holds for all $\phi \in C^\infty(Q)$ *with* $\nabla \cdot \phi = 0$ *and* $supp(\phi) \subset\subset Q \times [0,T)$.

Theorem 1.2 (Main Theorem)**.** *Let* $u_0 \in H$ *and* $f \in L^{q'}(Q)$. *Let* S *fulfill (I)–(IV). Furthermore, suppose*

$$2\frac{n+1}{n+2} < q < \infty. \tag{1.5}$$

Then, there exists a weak solutionb $u \in L^q(0,T;V_q) \cap C_w([0,T];H)$ *to* (1.1)–(1.4).

Strategy :

(1) First consider for bounded domains, and then take to infinity of the size of the domains
(2) Harmonic decomposition of L^r functions
(3) Monotone operator theory, Minty trick
(4) L^∞ truncation method

2. Harmonic decomposition of L^r functions

$G \subset \mathbb{R}^n$ be bounded domains with $\partial G \in C^2$.
$\|u\|_{W_0^{2,r}(G)} := \Big(\int_G |\Delta u|^r dx \Big)^{1/r}$

Lemma 2.1. $1 < r < \infty$,

$$\|u\|_{W_0^{2,r}(G)} \leq C_r \sup_{0 \neq W_0^{2,r'}(G)} \frac{\int_G \Delta u \Delta \phi dx}{\|\phi\|_{W_0^{2,r'}(G)}}, \quad \forall u \in W_0^{2,r}(G).$$

$$A^r(G) := \overline{\{\Delta\phi : \phi \in C_0^\infty(G)\}}^{L^r(G)},$$
$$B^r(G) := \{\varphi \in L^r(G) : \Delta\varphi = 0 \text{ in } G\}.$$

Lemma 2.2. *Let $p \in A^r(G)$ and $(h_{ij}) \in L^r(G)$, such that*

$$\int_G p\Delta\phi dx = \int_G h : \nabla^2\phi dx \;\forall \phi \in C_0^\infty(G).$$

Then,

$$\|p\|_{L^r(G)} \leq C\|h\|_{L^r(G)}.$$

Additionally, suppose $\nabla \cdot h \in L^\sigma(G)$ for $1 < \sigma < r$. Then $p \in W^{1,\sigma}loc(G)$ and $\forall G' \subset\subset G$, we have

$$\|p\|_{L^\sigma(G')} \leq C\big(\|\nabla \cdot h\|_{L^\sigma(G)} + \|h\|_{L^r(G)}\big).$$

Lemma 2.3. $1 < r < \infty$. *For* $v^* \in (W_0^{2,r'}(G))^*$, *there exists a unique* $u \in W_0^{2,r}(G)$ *such that*

$$\int_G \Delta u \Delta \phi dx = \langle v^*, \phi \rangle \quad \forall \phi \in C_0^\infty(G).$$

Corollary 2.4 (Simader)**.** *For* $p \in L^r(G)$, *there exist* $p_0 \in A^r(G)$ *and* $p_h \in B^r(G)$ *such that*

$$p = p_0 + p_h.$$

In addition, we have

$$\|p_0\|_{L^r(G)} + \|p_h\|_{L^r(G)} \le C\|p\|_{L^r(G)}.$$

Theorem 2.5. *Let* $Q \in L^r(G \times (0,T))^{n^2}$ *and* $u \in C_w([0,T]; L^2(G))$ *with* $\nabla \cdot u = 0$. *Suppose that*

$$-\int_0^T \int_G u \cdot \partial_t \varphi dx dt + \int_0^T \int_G Q : \nabla \varphi dx dt = 0$$

holds for $\varphi \in C_0^\infty(G \times (0,T))$ *with* $\nabla \cdot \varphi = 0$.

Then, there exist unique functions $p_0 \in L^r(0,T; A^r(G))$ *and* $\tilde{p}_h \in C_w([0,T]; \dot{B}^r(G))$, *such that*

$$\begin{aligned} -\int_0^T \int_G u \cdot \partial_t \varphi dx dt &+ \int_0^T \int_G Q : \nabla \varphi dx dt \\ &= \int_0^T \int_G p_0 \nabla \cdot \varphi dx dt - \int_0^T \int_G \tilde{p}_h \partial_t \nabla \cdot \varphi dx dt + \int_G u(0) \cdot \varphi(0) dx \end{aligned}$$

for all $\varphi \in C^\infty(G \times (0,T))$ *with supp*$(\varphi) \subset\subset G \times [0,T)$. *In addition, we have a priori estimates*

$$\|p_0\|_{L^r(G\times(0,T))} \leq c\|Q\|_{L^r(G\times(0,T))},$$
$$\|\tilde{p}_h\|_{L^\infty(0,T;L^2(G))} \leq c\|u\|_{L^\infty(0,T;L^r(G))} + c\|Q\|_{L^r(G\times(0,T))},$$

3. Existence of weak solutions to the approximate system

Let $\Phi \in C^\infty([0,\infty))$ be a non decreasing function, with $0 \leq \Phi \leq 1$, $\Phi \equiv 1$ on $[0,1]$, $\Phi \equiv 0$ in $[2,\infty)$, and $0 \leq -\Phi' \leq 2$.

Consider the approximate system:

$$\partial_t u_\varepsilon + \nabla \cdot \big(u_\varepsilon \otimes u_\varepsilon \Phi_\varepsilon(|u_\varepsilon|)) - S(x,t,D(u_\varepsilon)) + f\big) = -\nabla p_\varepsilon in Q, \tag{3.1}$$
$$\nabla \cdot u_\varepsilon = 0 in Q, \tag{3.2}$$
$$u_\varepsilon\big|_{\partial\Omega\times(0,T)} = 0, \tag{3.3}$$
$$u_\varepsilon(0) = u_0 in \Omega. \tag{3.4}$$

(1) Ω bounded, with Schauder fixed point theorem, we get local existence
(2) extend arbitrarily large interval to get global weak solution
(3) Ω unbounded, Ω is a union of bounded open sets.

4. Proof of Main Theorem

Take $\epsilon \to 0$.

We have weak convergence

$$\begin{aligned} u_{epsilon_m} &\to u \text{ weakly in } L^q(0,T;V_q), \\ u_{epsilon_m} &\to u \text{ weakly in } L^{q\frac{n+2}{n}}(Q), \\ S(\cdot, D(u_{epsilon_m})) &\to \tilde{S} \text{ weakly in } L^{q'}(Q), \\ u_{epsilon_m} \otimes u_{\epsilon_m} &\to \tilde{H} \text{ weakly in } L^{q\frac{n+2}{2n}}(Q). \end{aligned}$$

Aubin-Lyons compact embedding lemma

Need Harmonic decomposition and Cacciopoli inequality for $\tilde{p}_{h,\epsilon}$, Lebesgue dominated convergence theorem

Vitali theorem

Monotonicity of S, Minty's trick,

to apply monotone operator theorem we need the convection term belonging to L^σ, $\sigma > 1$. Then we need that

$$\frac{1}{q} + \frac{n}{q(n+2)} < 1 \iff q > 2\frac{n+1}{n+2}.$$

Therefore, we need the strong convergence of ∇u^m in $L^s(Q_T)^{3\times 3}$ for some $s \geq 1$.