Mathematical Theory of Non-Newtonian Fluid

1. Derivation of the Incompressible Fluid Dynamics

2. Existence of Non-Newtonian Flow and its Dynamics

3. Existence in the Domain with Boundary

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Drivation of the Incompressible Fluid Equations

1. Introduction

2. Fluids

3. Navier–Stokes Equations

4. History for Navier−Stokes Equations

5. Non−Newtonian Fluids

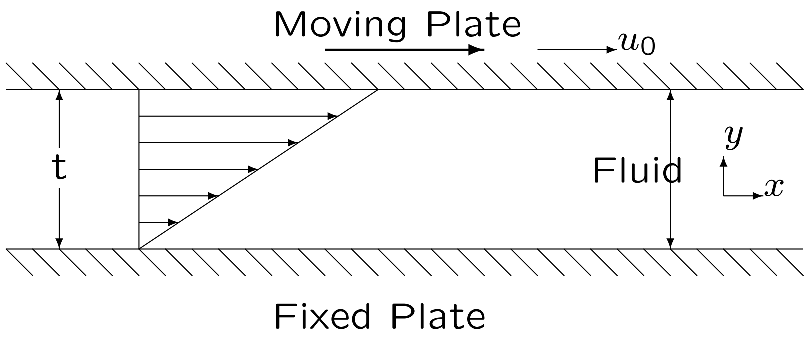
Introduction

Fluids: liquids and gases

A solid can resist an applied shear force and remain at rest, while a fluid cannot.

Newtonian Fluid : water, air

Non−Newtonian Fluid: grease, polymer, biological fluid



Velocity distribution between two parallel plate

Properties of Fluids:

1. Kinematic property
   −− linear **velocity**, angular velocity, vorticity, accerlation, and strain rate

2. Transport property
   −− **viscosity**, termal conductivity, mass diffusivity

3. Thermodynamic property
   −− **pressure**, thermal conductivity, temperature, enthalpy, entropy, specific heat

4. Other miscellaneous property
   −− surface tension, vapor pressure

Kinematic Property:

- Lagrangean description : the scheme of following the trajectories of indivisual particles

- Eulerian description : the scheme describing the flow at fixed point as a function of time

Eulerian velocity vector fields

$$\begin{aligned}\mathbf{u}(\mathbf{x},t) &= \mathbf{u}(x_1,x_2,x_3,t)\\ &= (u_1(x_1,x_2,x_3,t),u_2(x_1,x_2,x_3,t),u_3(x_1,x_2,x_3,t))\end{aligned}$$

Three fundamental law of mechanics :

Conservation of **mass, momentum**, energy in Lagrangean sense

[Relationship of Lagrangean and Eulerian Coordinates]

Let Q be any property

$$dQ = \frac{\partial Q}{\partial x_1} dx_1 + \frac{\partial Q}{\partial x_2} dx_2 + \frac{\partial Q}{\partial x_3} dx_3 + \frac{\partial Q}{\partial t} dt.$$

Since

$$dx_1 = u_1 dt, \qquad dx_2 = u_2 dt, \qquad dx_3 = u_3 dt,$$

we have

$$\begin{aligned} \frac{dQ}{dt} &= \frac{\partial Q}{\partial t} + u_1 \frac{\partial Q}{\partial x_1} + u_2 \frac{\partial Q}{\partial x_2} + u_3 \frac{\partial Q}{\partial x_3} \\ &= \frac{\partial Q}{\partial t} + (\mathbf{u} \cdot \nabla) Q \end{aligned}$$

$$\begin{cases} \frac{DQ}{Dt} = \frac{dQ}{dt} & -- \text{ material derivative} \\ (\mathbf{u} \cdot \nabla) Q & -- \text{ convective derivative} \\ \frac{\partial Q}{\partial t} & -- \text{ local derivative} \end{cases}$$

[Reynolds' Transport Theorem]

For a control volume,

$$\begin{aligned}
&\frac{D}{Dt}\int_{V(t)} Q(t)dV \\
&= \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)} Q(t+\delta t)dV - \int_{V(t)} Q(t)dV\right\}\right. \\
&= \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)-V(t)} Q(t+\delta t)dV\right\}\right. + \int_{\partial V(t)} \frac{\partial Q(t)}{\partial t}dV \\
&= \lim_{\delta t\to 0}\int_{\partial V(t)} Q(t+\delta t)\mathbf{u}\cdot n dS + \int_{V(t)}\frac{\partial Q}{\partial t}dV \\
&= \int_{\partial V(t)} Q(t)\mathbf{u}\cdot n dS + \int_{V(t)}\frac{\partial Q}{\partial t}dV \\
&= \int_{V(t)}\left[\nabla\cdot(Q\mathbf{u}) + \frac{\partial Q}{\partial t}\right]dV.
\end{aligned}$$

Motion of Fluid particle:

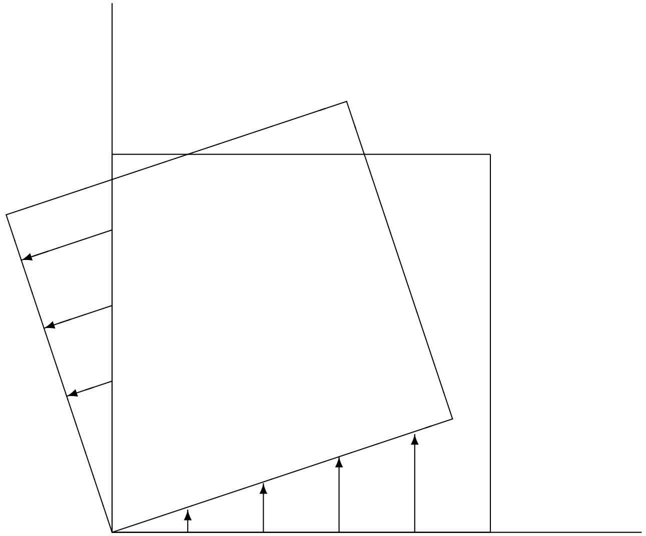
- translation – $\mathbf{u}(\mathbf{x}, t)$
- rotation – $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$
- deformation
  1. dilation : expansion or reduction of volume
  $$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial w}{\partial z}$$
  2. shear strain : distortion
  – the average decrease of the angle between two lines which are initially perpendicular in the unstrained state
  $$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Notice that

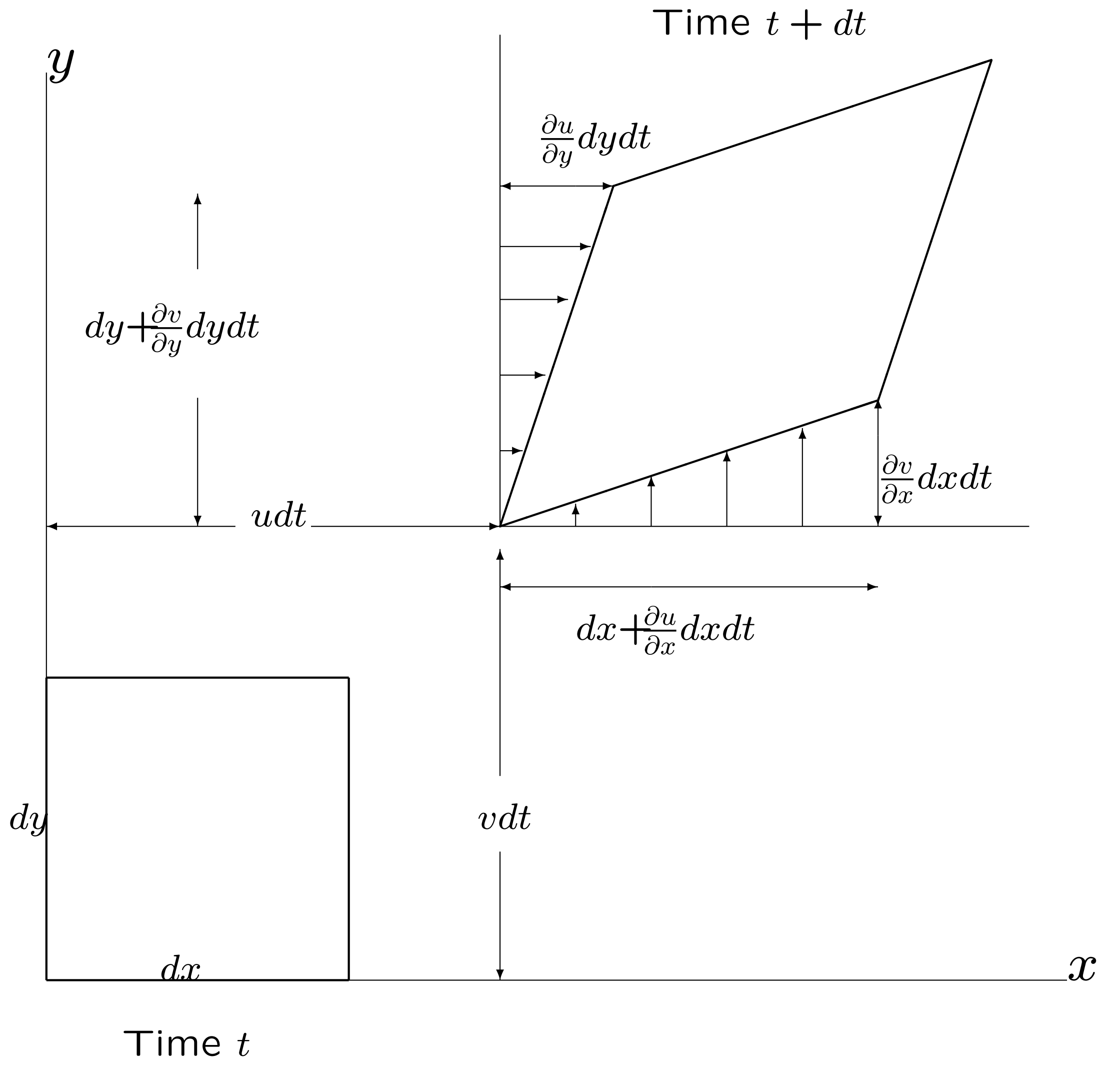
$$u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{i,j} - u_{j,i}).$$

Means : each velocity derivative can be resolved into a **strain rate** plus an **angular velocity**

Rotation :

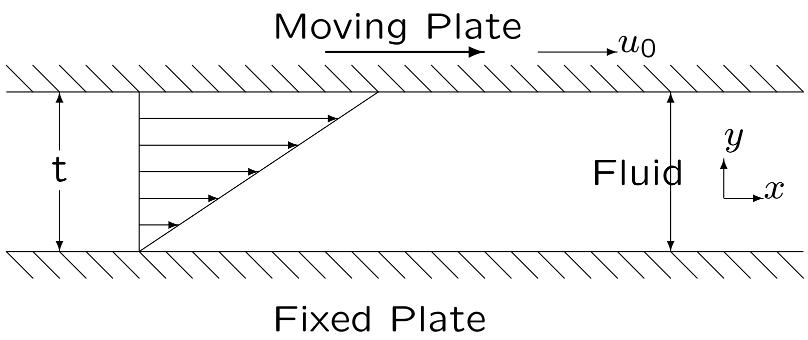


Distortion of a moving fluid element:



Transport property :

Viscosity : the property of a fluid which relates applied **stress** to the resulting **straing rate**



By the viscosity property of a fluid, if the upper plate move with speed V, then the fluid on the plate move with the same speed V.

Since there is no movement on fixed plate, the fluid on the ground doee not move.

$$[\text{Shear strain} = \epsilon_{ij}] = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\frac{du}{dy} = \tau_{xy}$$

In general, shear stress τ_{ij} is a function of the strain rate ϵ_{ij}:

$$\tau_{ij} = f(\epsilon_{ij})$$

- If the relation is linear, it is Newtonian;

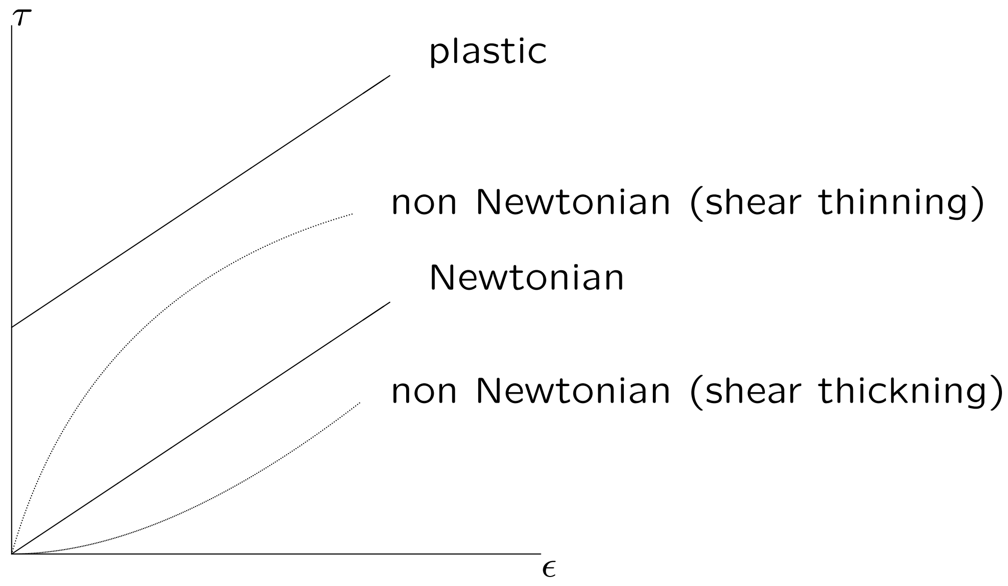
$$\tau_{ij} = \mu\frac{\partial u}{\partial y}.$$

  The constant μ is the **viscosity**.

- If not, it is non-Newtonian.

$$\tau_{xy} = K\epsilon_{xy}^{r}$$

- τ_{xy} : shear stress,
- $\frac{\partial u}{\partial y}$: rate of strain
- μ : viscosity



τ
plastic
non Newtonian (shear thinning)
Newtonian
non Newtonian (shear thickning)
ϵ

Navier-Stokes equation

1. Conservation of Mass : Equation of Continuity

$$
\begin{aligned}
0 &= \frac{D}{Dt}\int_V \rho dV \\
&= \int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k)\right] dV
\end{aligned}
$$

$\Leftarrow$ Reynold's transport theorem

ρ: density

Therefore,

$$
0 = \frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{u}) = \frac{D\rho}{Dt} + \rho \mathrm{div}\mathbf{u}.
$$

If $\mathrm{div}\,\mathbf{u} = 0$, then the fluid is called **incompressible**.

2. Conservation of Momentum (**F**=m**a**):

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}$$

$$\begin{aligned}
&\frac{D}{Dt}\int_V \rho \mathbf{u} dV = \text{[total force]} \\
&= \text{[body force]} + \text{[surface (exterior) force]} \\
&= \int_V \rho F_i dV + \int_S \tau_{ij} n_j dS \\
&= \int_V \left(\rho F_i + \frac{\partial \tau_{ij}}{\partial x_j}\right) dV \quad \Longleftarrow \text{[divergence theorem]}
\end{aligned}$$

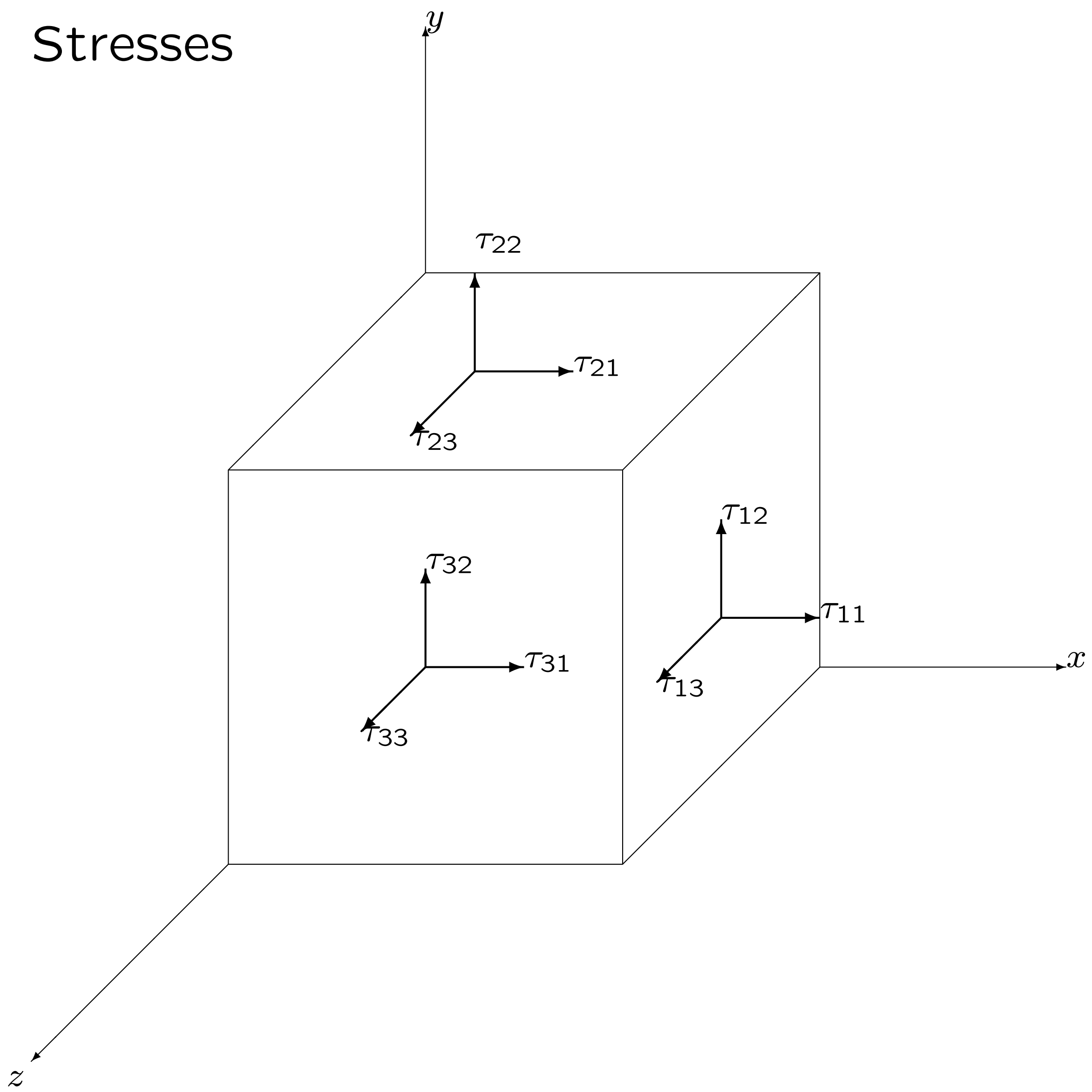
$\Longrightarrow$

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

- Body forace $F_i = \mathbf{g} \Longleftarrow$ external fields such as gravity or electromagnetic potential

- Surface force div$\tau \Longleftarrow$ external stress on the sides of the particle

Stresses



τ_{ij} : stress in the j direction on a face normal to the i axis

[Relation of force and deformation]

∘ fluid at rest :

$$\tau_{ij} = -\rho\delta_{ij}$$

$-p = \tau_{xx} = \tau_{yy} = \tau_{zz}$: normal stress = pressure

∘ fluid in motion:

- Newtonian fluid: linear relation of τ_{ij} and deformation rate

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda \text{div}\mathbf{u}$$

- Newtonian fluid: non-linear relation of τ_{ij} and deformation rate

Navier-Stokes Equations: Incompressible and Newtonian Fluid

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{u}$$

$\Longrightarrow$

$$\begin{cases} \dfrac{du_i}{dt} - \nu \Delta u_i + (u \cdot \nabla) u_i + \nabla_i p = f_i, \\ \\ \nabla \cdot u = 0 \end{cases}$$

- Leray, 1934 Hopf, 1951
  Existence of Weak Solutions

- 2D N-S: Uniquness and Regularity

$$u \in L^\infty(0,\infty; H^1) \cap L^2(0,\infty; H^2)$$

- 3D N-S: Short Time Regularity

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$$

- 3D N-S: Small Data, or Large Viscosity
$\Longrightarrow$ Global Regularity

- 3D N-S: Regularity and Uniqueness :
qquad Open Problem

Regularity

- Serrin, 1962

$$L^{r,r'}, \qquad \frac{3}{r} + \frac{2}{r'} < 1.$$

- Caffarelli,Kohn & Nirenberg, 1982

- Constantin & Fefferman

- Choe, 1996 : Boundary L^∞ Regularity

Regularity Theorem for Navier-Stokes Equations[Bae and Choe, 1996, 2007]

Suppose that (u, p) is a weak solution. Let $u = (u_1, u_2, u_3)$. If $u_1, u_2 \in L^{\infty,\infty}_{loc}(Q)$, then

$$u_3 \in L^{\infty,\infty}_{loc}(Q).$$

Therefore, u is smooth.

Our Plan

- Use $\int \int |u|^3 \; dx \; dt \leq C$

- Show $\sup \int |u|^3 \; dx \leq C$

- Show $\int \int_{loc} |u| \; |\nabla u|^2 \; dx \; dt \leq C$

- Show $u \in L^{5,5}$

- Use Struwe's Regularity: If $u \in L^{5,5}$ Then u regular

Existence and Regularity Theorem for Non-Newtonian Fluids[Bae and Choe, 1997]

$$\begin{cases} \dfrac{du_i}{dt} - \nu_1 \Delta u_i - \nu_2 \dfrac{\partial |e_{ij}|^r \epsilon_{ij}}{\partial x_j} + (u \cdot \nabla) u_i + \nabla_i p = f_i, \\ \\ \nabla \cdot u = 0 \\ \epsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial_j} + \frac{\partial u_j}{\partial x_i} \Big) \end{cases}$$

Then thereis a weak solution for $r > -1$.

For $r > 2/5$ the solution is regular.