# FIRST-ORDER SYSTEM LEAST-SQUARES METHOD FOR THE OPTIMAL CONTROL BY THE STOKES EQUATIONS

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## ABSTRACT

We consider the approximate solution of optimal control problem governed by the Stokes equations. For the constrained optimization problem, Lagrange multiplier method is used. We employ a least-squares functional developed in [3] for the two- and three-dimensional first order optimality systems, by introducing a *velocity flux* variable and associated curl and *trace* equations. With the provided $H^2$-regularity of that system, ellipticity in an $H^1$ product norm is established. This yields optimal discretization error estimates for finite element spaces in a product $H^1$ norm. The resulting matrix becomes symmetric and positive definite. For numerical tests, we apply V-cycle multigrid methods to the total discrete algebraic system.

## COUPLED FIRST-ORDER SYSTEM FORMULATIONS

Let $\Omega \subset \mathbb{R}^n (n = 2 \text{ or } 3)$ be an open, connected and bounded domain with Lipschitz boundary $\partial\Omega$. We consider the control problem consisting of minimizing the quadratic functional defined by

$$\min \mathcal{J}(\mathbf{u}, \mathbf{f}) = \frac{1}{2}\int_\Omega |\mathbf{u} - \hat{\mathbf{u}}|^2 \, dx + \frac{\sigma}{2}\int_\Omega |\mathbf{f}|^2 \, dx \tag{1}$$

subject to the Stokes equation

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial\Omega \\ \int_\Omega p \, dx = 0, & \end{cases} \tag{2}$$

where $\sigma$ is positive constant.

Then, the constrained optimization problem (1)-(2) is the equivalent to the unconstrained optimization problem of finding saddle points of the Lagrangian functional. These saddle points may be found by solving the optimality system:

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \\ \nu\Delta\mathbf{v} + \nabla q + \mathbf{u} = \hat{\mathbf{u}} & \text{in} \quad \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad \Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in} \quad \Omega \\ -\sigma\mathbf{f} = \mathbf{v} & \text{in} \quad \Omega. \end{cases} \tag{3}$$

According to [3], This problem can be rewritten as the first order system of partial differential equations by introducing the velocity flux variable $\underline{\mathbf{U}} = \nabla\mathbf{u}^t = (\nabla u_1, \nabla u_2, \ldots, \nabla u_n)$, and $\underline{\mathbf{V}} = \nabla\mathbf{v}^t = (\nabla v_1, \nabla v_2, \ldots, \nabla v_n)$, (3) can be written as an extended first order system of partial differential equations:

$$\begin{cases} -\nu(\nabla\cdot\underline{\mathbf{U}})^t + \nabla p + \dfrac{\mathbf{v}}{\sigma} = \mathbf{0} & \text{in}\,\Omega \\ \underline{\mathbf{U}} - \nabla\mathbf{u}^t = \underline{\mathbf{0}} & \text{in}\,\Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in}\,\Omega \\ \nabla\times\underline{\mathbf{U}} = \mathbf{0} & \text{in}\,\Omega \\ \nabla(\text{tr}\underline{\mathbf{U}}) = \mathbf{0} & \text{in}\,\Omega \\ \mathbf{u} = \mathbf{0} & \text{on}\,\Gamma \\ \mathbf{n}\times\underline{\mathbf{U}} = \mathbf{0} & \text{on}\,\Gamma, \end{cases} \tag{4}$$

$$\begin{cases} \nu(\nabla\cdot\underline{\mathbf{V}})^t + \nabla q + \mathbf{u} = \hat{\mathbf{u}} & \text{in}\,\Omega \\ \underline{\mathbf{V}} - \nabla\mathbf{v}^t = \underline{\mathbf{0}} & \text{in}\,\Omega \\ \nabla\cdot\mathbf{v} = 0 & \text{in}\,\Omega \\ \nabla\times\underline{\mathbf{V}} = \mathbf{0} & \text{in}\,\Omega \\ \nabla(\text{tr}\underline{\mathbf{V}}) = \mathbf{0} & \text{in}\,\Omega \\ \mathbf{v} = \mathbf{0} & \text{on}\,\Gamma \\ \mathbf{n}\times\underline{\mathbf{V}} = \mathbf{0} & \text{on}\,\Gamma, \end{cases} \tag{5}$$

Then, we may make a least-squares functional as in [3], which are

$$\begin{aligned} G(&\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q;\hat{\mathbf{u}}) \\ &= \|\nu(\nabla\cdot\underline{\mathbf{U}})^t - \nabla p - \frac{\mathbf{v}}{\sigma}\|^2 + \|\underline{\mathbf{U}} - \nabla\mathbf{u}^t\|^2 + \|\nabla\cdot\mathbf{u}\|^2 + \|\nabla\times\underline{\mathbf{U}}\|^2 \\ &+ \|\nabla\text{tr}(\underline{\mathbf{U}})\|^2 + \|\nu(\nabla\cdot\underline{\mathbf{V}})^t + \nabla q + \mathbf{u} - \hat{\mathbf{u}}\|^2 + \|\underline{\mathbf{V}} - \nabla\mathbf{v}^t\|^2 \\ &\quad + \|\nabla\cdot\mathbf{v}\|^2 + \|\nabla\times\underline{\mathbf{V}}\|^2 + \|\nabla\text{tr}(\underline{\mathbf{V}})\|^2. \end{aligned} \tag{6}$$

Hence we are going to minimize $G(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q;\hat{\mathbf{u}})$. Since the coercivity and continuity of the bilinear form corresponding to $G$ are equivalent to stating that the functional $G$ is norm-equivalent, the following theorems show the existence and uniqueness of the minimizer.

**Theorem 1.** Assume that the domain $\Omega$ is a bounded convex polyhedron or has $C^{1,1}$ boundary. Then there are two constants $C_1$ and $C_2$ dependent on $\sigma$ and $\nu$ such that for any $(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q) \in \mathcal{W}\times\mathcal{W}$, we have

$$C_1\, M(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q) \le G(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q;\mathbf{0}), \tag{7}$$

and

$$G(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q;\mathbf{0}) \le C_2\, M(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q), \tag{8}$$

where

$$M(\underline{\mathbf{U}},\mathbf{u},p,\underline{\mathbf{V}},\mathbf{v},q) = \|\underline{\mathbf{U}}\|_1^2 + \|\mathbf{u}\|_1^2 + \|p\|_1^2 + \|\underline{\mathbf{V}}\|_1^2 + \|\mathbf{v}\|_1^2 + \|q\|_1^2.$$

The discrete problem for minimizing (6) is equivalent to an linear algebraic system:

$$\begin{bmatrix} \mathbb{K}_1 & \mathbb{C}_1^t & \mathbb{C}_2^t & \mathbf{0} & \mathbb{C}_4^t & \mathbf{0} \\ \mathbb{C}_1 & \mathbb{K}_2 & \mathbf{0} & \mathbb{C}_3^t & \mathbf{0} & \mathbb{C}_6^t \\ \mathbb{C}_2 & \mathbf{0} & \mathbb{K}_3 & \mathbf{0} & \mathbb{C}_5^t & \mathbf{0} \\ \mathbf{0} & \mathbb{C}_3 & \mathbf{0} & \frac{1}{\sigma^2}\mathbb{K}_1 & \frac{1}{\sigma^2}\mathbb{C}_1^t & -\frac{1}{\sigma^2}\mathbb{C}_2^t \\ \mathbb{C}_4 & \mathbf{0} & \mathbb{C}_5 & \frac{1}{\sigma^2}\mathbb{C}_1 & \mathbb{K}_4 & \mathbf{0} \\ \mathbf{0} & \mathbb{C}_6 & \mathbf{0} & -\frac{1}{\sigma^2}\mathbb{C}_2 & \mathbf{0} & \frac{1}{\sigma^2}\mathbb{K}_3 \end{bmatrix} \begin{bmatrix} \vec{U} \\ \vec{u} \\ \vec{p} \\ \vec{V} \\ \vec{v} \\ \vec{q} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{g}_1 \\ \vec{0} \\ \vec{g}_2 \\ \vec{0} \\ \vec{g}_3 \end{bmatrix}, \tag{9}$$

where $\vec{U} = (U_1,\ldots,U_{4J})^t$, $\vec{u} = (u_1,\ldots,u_{2J})^t$, $\vec{p} = (p_1,\ldots,p_J)^t$, $\vec{V} = (V_1,\ldots,V_{4J})^t$, $\vec{v} = (v_1,\ldots,v_{2J})^t$, $\vec{q} = (p_1,\ldots,q_J)^t$. The block matrices $\mathbb{K}_1$ and $\mathbb{K}_4$ are of size $4J \times 4J$, $\mathbb{K}_2$ is of size $2J \times 2J$, $\mathbb{K}_3$ is of size $J \times J$, $\mathbb{C}_1$ and $\mathbb{C}_4$ are of size $2J \times 4J$, $\mathbb{C}_2$ is of size $J \times 4J$, $\mathbb{C}_3$ is of size $4J \times 2J$, $\mathbb{C}_5$ is of size $2J \times J$ and $\mathbb{C}_6$ is of size $J \times 2J$.

## NUMERICAL EXPERIMENTATIONS WITH MULTIGRID METHOD

Table 1 The $L^2$-errors $\|\mathbf{u}^h - \hat{\mathbf{u}}\|$ between the controlled flow $\mathbf{u}^h$ and the target flow $\hat{\mathbf{u}}$, the $L^2$-norm of control $\|\mathbf{f}^h\|$ and the values of the cost functional for the different $\sigma$ when $h = 1/32$.

| $\nu$ | $\sigma$ | $\|\mathbf{u}^h - \hat{\mathbf{u}}\|$ | $\|\mathbf{f}^h\|$ | $\mathcal{J}(\mathbf{u}^h, \mathbf{f}^h)$ |
|---|---|---|---|---|
| 1 | 1 | 1.273898834e-1 | 2.334794985e-3 | 8.116816833e-3 |
| | $10^{-1}$ | 1.239166887e-1 | 1.890008890e-2 | 7.695533536e-3 |
| | $10^{-2}$ | 3.700037256e-2 | 5.435863697e-2 | 6.992880920e-4 |
| | $10^{-3}$ | 7.202576194e-4 | 8.281429727e-3 | 2.936765583e-7 |
| | $10^{-4}$ | 6.576050304e-4 | 7.602196141e-2 | 5.051891188e-7 |
| $10^{-1}$ | 1 | 1.210256866e-1 | 2.213912607e-2 | 7.568678854e-3 |
| | $10^{-1}$ | 3.584283083e-2 | 6.274538301e-2 | 8.392034154e-4 |
| | $10^{-2}$ | 1.078722145e-3 | 3.168275641e-2 | 5.600806001e-6 |
| | $10^{-3}$ | 5.974841291e-4 | 4.485327213e-2 | 1.184401653e-6 |
| | $10^{-4}$ | 6.145601493e-4 | 6.459243032e-1 | 2.104975236e-5 |
| $10^{-2}$ | 1 | 2.323299951e-2 | 3.816631056e-2 | 9.982197639e-4 |
| | $10^{-1}$ | 6.127488652e-4 | 2.493789498e-2 | 3.128266088e-5 |
| | $10^{-2}$ | 5.433031180e-4 | 1.010494889e-2 | 6.581390993e-7 |
| | $10^{-3}$ | 5.523628405e-4 | 5.717007883e-2 | 1.786761311e-6 |
| | $10^{-4}$ | 5.532245028e-4 | 7.030575325e-1 | 2.486752338e-5 |

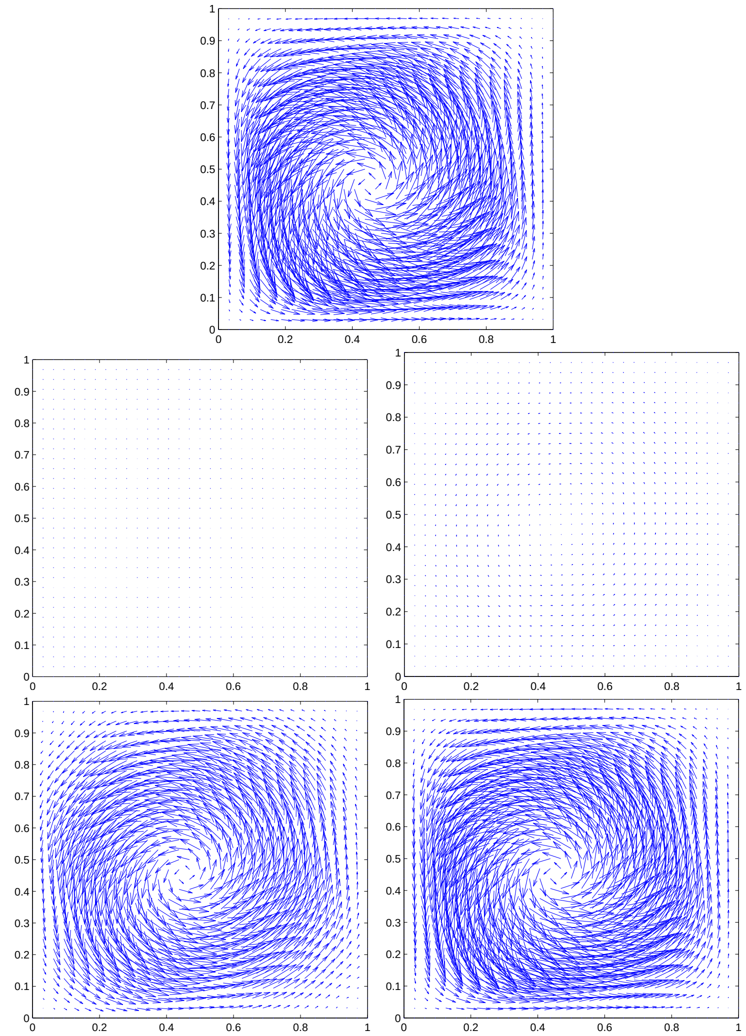

Figure 1. Target flow(top) and controlled flows at $\sigma = 1$(middle left),$0.1$(middle right), $0.01$(bottom left) and $\sigma = 0.001$(bottom right) for $\nu = 1$, when $h = 1/32$.


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