Optimal Error Estimate for Semi-Discrete Gauge-Uzawa Method for the Navier-Stokes Equations

Jae-Hong Pyo [1]

1) *Department of Mathematics, Kangwon National University, Chuncheon, Korea.*

Corresponding Author : Jae-Hong Pyo, jhpyo@kangwon.ac.kr

ABSTRACT

The Gauge-Uzawa finite element method (GU-FEM)[3] is a fully discrete projection type method to solve the evolution Navier-Stokes equations, which overcomes many shortcomings of projection methods and displays superior numerical performance. Since Gauge-Uzawa is consistent with Navier-Stokes equations, it can be applied easily to more complicate fluid problems [4,6] and the normal mode solution shows full accuracy without spurious boundary layer term for smooth solutions in contrast other projection type methods [7]. In addition, it is an unconditionally stable scheme [3]. However, we have only suboptimal accuracy via the energy estimate which has been proved in [3]. In this paper, we consider semi-discrete Gauge-Uzawa method to prove optimal convergence via energy estimate: $\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_0 \le C\tau$ and a distinctive result $\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_1 + \|p(t^{n+1}) - p^{n+1}\|_0 \le C\tau$.

GAUGE-UZAWA METHOD

Gauge Uzawa [3,5] to solve the Navier-Stokes equations can be written as following: Let $\psi(\mathbf{x})$ be any scalar function. Start with initial values $s^0 = 0$ and $\mathbf{u}^0 = \mathbf{u}(0,\mathbf{x})$. Repeat for $1 \le n \le N$,

Step 1 Find $\widehat{\mathbf{u}}^{n+1}$ as the solution of

$$\begin{aligned} \frac{\widehat{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\tau} + (\mathbf{u}^n \cdot \nabla)\widehat{\mathbf{u}}^{n+1} + \mu\nabla s^n - \mu\triangle\widehat{\mathbf{u}}^{n+1} &= \mathbf{f}(t^{n+1}) - \mu\nabla\triangle\psi, \text{ in } \Omega, \\ \widehat{\mathbf{u}}^{n+1} &= \mathbf{0}, \qquad \text{on } \partial\Omega, \end{aligned}$$

Step 2 Find ρ^{n+1} as the solution of

$$\begin{aligned} -\triangle\rho^{n+1} &= \text{div } \widehat{\mathbf{u}}^{n+1}, \quad \text{in } \Omega, \\ \partial_{\boldsymbol{\nu}}\rho^{n+1} &= 0, \qquad \text{on } \partial\Omega, \end{aligned}$$

Step 3 Update $\mathbf{u}^{n+1}$ and s^{n+1}

$$\begin{aligned} \mathbf{u}^{n+1} &= \widehat{\mathbf{u}}^{n+1} + \nabla\rho^{n+1}, \\ s^{n+1} &= s^n - \text{div } \widehat{\mathbf{u}}^{n+1}. \end{aligned}$$

One may compute the pressure whenever necessary as

$$p^{n+1} = -\frac{\rho^{n+1}}{\tau} + \mu s^{n+1} + \mu\triangle\psi.$$

PREVIOUS RESULTS

We have proved in [3] suboptimal convergence results:

$$\tau \sum_{n=0}^{N} \left\| \nabla \left(\mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right) \right\|_0^2 \leq C\tau,$$

$$\tau \sum_{n=0}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_0^2 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_0^2 \right) \leq C\tau^2,$$

and for pressure and time-derivative of velocity;

$$\tau \sum_{n=0}^{N} \left(\left\| \delta\mathbf{u}(t^{n+1}) - \delta\mathbf{u}^{n+1} \right\|_0^2 + \left\| p(t^{n+1}) - p^{n+1} \right\|_0^2 \right) \leq C\tau,$$

where δ is the discrete time derivative for a sequence $\{W^n\}_{n=0}^N$ to be

$$\delta W^{n+1} := \frac{W^{n+1} - W^n}{\tau}.$$

THE MAIN RESULTS

In this work, we obtain newly optimal convergence results:

$$\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_0 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_0 \leq C\tau,$$

$$\tau \sum_{n=0}^{N} \left(\left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_1^2 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_1^2 \right) \leq C\tau^2,$$

$$\left\| \delta\mathbf{u}(t^{n+1}) - \delta\mathbf{u}^{n+1} \right\|_0 + \left\| \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1} \right\|_1 \leq C\tau,$$

$$\left\| \delta\mathbf{u}(t^{n+1}) - \delta\widehat{\mathbf{u}}^{n+1} \right\|_0 + \left\| \mathbf{u}(t^{n+1}) - \widehat{\mathbf{u}}^{n+1} \right\|_1 \leq C\tau.$$

Additionally, the error of pressure becomes

$$\left\| p(t^{n+1}) - p^{n+1} \right\|_0 \leq C\tau.$$

REFERENCES

1. R. Nochetto and J.-H. Pyo, “Optimal relaxation parameter for the Uzawa method”, *Numer. Math.*, vol. 98, 2004, pp. 695 – 702.
2. R. Nochetto and J.-H. Pyo, “Error estimates for semi-discrete gauge methods for the evolution Navier-Stokes equations”, *Math. Comp.*, Vol. 74, 2005, pp. 521–542.
3. R. Nochetto and J.-H. Pyo, “ A finite element Gauge-Uzawa method. Part I : the Navier-Stokes equations” *SIAM J. Numer. Anal.*, Vol. 43, 2005, pp. 1043–1068.
4. R. Nochetto and J.-H. Pyo, “A finite element Gauge-Uzawa method. Part II : the Boussinesq equations” *Math. Models Methods Appl. Sci.*, 16, (2006), 1599–1626.

5. J.-H. Pyo, “The Gauge-Uzawa and related projection finite element methods for the evolution Navier-Stokes equations”, *Ph.D dissertation, University of Maryland* 2002.
6. J.-H. Pyo and J. Shen, “Gauge Uzawa methods for incompressible flows with variable density”, *J. Comput. Phys.*, 211, (2007), 181–197.
7. J.-H. Pyo and J. Shen, “ Normal mode analysis for a class of 2nd-order projection type methods for unsteady Navier-Stokes equations”, *Discrete Contin. Dyn. Syst. Ser. B*, Vol. 5, 2005, pp. 817–840.