AN ITERATIVE SUBSTRUCTURING METHOD WITH LAGRANGE MULTIPLIERS

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ABSTRACT

We consider the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ and f is a given function in $L^2(\Omega)$. For the sake of simplicity, we assume that Ω is partitioned into two subdomains $(\Omega_i)_{i=1,2}$ such that $\overline{\Omega} = \bigcup_{i=1}^{2} \overline{\Omega}_i$ and $\Omega_1 \bigcap \Omega_2 = \emptyset$. The problem (1) can be rewritten as

$$\min_{v \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega \nabla v \cdot \nabla v \, dx - \int_\Omega f v \, dx \right),$$

equivalently

$$\min_{\substack{v_i \,\in\, H^1(\Omega_i) \\ v_i \,=\, 0 \text{ on } \partial\Omega \\ v_1 \,=\, v_2 \text{ on } \partial\Omega_1 \cap \partial\Omega_2}} \quad \sum_{i=1}^{2} \left(\frac{1}{2} \int_{\Omega_i} \nabla v_i \cdot \nabla v_i \, dx - \int_{\Omega_i} f v_i \, dx \right). \tag{2}$$

In the domain-decomposition approach based on the reformulated minimization problem (2) with a constraint, a critical point is how to convert a constrained minimization problem into a unconstrained one. Most studies [1–3] for treatment of constrained minimizations started in the field of optimal control problem. There are the three most popular methods developed for different purposes; the Lagrangian method, the method of penalty function, and the method of multipliers (the augmented Lagrangian method). Such various ideas have been introduced for handling constraints as the continuity across the interface in (2) (see [4–6]).

In this talk, we present a dual iterative substructuring algorithm for the second order elliptic problem which deals with the continuity constraint across the interface in view of the augmented Lagrangian method. Instead of the typical energy functional in (2), we start with a penalized energy functional by augmenting a quadratic penalty term with a positive parameter η which measures the jump across the interface. With keeping the continuity constraint, we consider the minimization of the penalized functional. Introduction of Lagrange multipliers to impose the continuity constraint gives a saddle-point problem related to the augmented Lagrangian functional. Then, we derive a dual system with Lagrange multipliers as unknowns from the saddle-point problem and solve it iteratively by the conjugate gradient method (CGM). For $\eta = 0$, the proposed algorithm is reduced to the FETI-DP method which is one of the most

advanced dual substructuring methods. Since the penalty term plays a role in making the jump smaller as η increases, it is obvious that the addition of the penalty term is redundant to the continuity constraint. Based on the fact, it is easily noted that the solution of the designed algorithm restores precisely the solution from the FETI-DP formulation. In fact, such a combination of penalization and dualization is purposed to have the best of both worlds. Unlike penalty methods, the convergence of solution is guaranteed without making η tend to infinity. On the other hand, the added penalty term contributes towards enhancing the convergence speed in view of iterative implementation.

When focusing on the iterative routine of CGM, an important indicator of the efficiency of a domain decomposition method is the bound on the growth of the condition number of the relevant system when the mesh is refined and the number of subdomains increases. We prove that the proposed method is scalable in the sense that for large values of η, the condition number of the dual system has a constant bound independent of the subdomain size H and the element size h, unlike most efficient iterative substructuring methods including FETI-DP where the logarithmic growth $(1+\log(H/h))^2$ in the condition number is observed. To the best of our knowledge, the algorithm with such a constant bound of condition number is unprecedented according to previous studies in the field of domain-decomposition approach.

In addition, we treat some computational issues in practical sense. An extremely large value of η is not necessary for either improvement of accuracy or speed-up of iterative solver. But, in order to avoid trouble in choosing a properly large η, we check how the proposed method behaves in practice as η increases. Unfortunately, in the process of CGM implementation, we need to solve a system which becomes more ill-conditioned in proportion to increment of η. To cope with ill-conditioning behavior, a optimal preconditioner with respect to η is developed. The proposed method is compared with FETI-DP for evaluation of its practical performance in terms of total CPU time. According to computational results, the presented method is superior to FETI-DP. In addition, it is comparable to the preconditioned FETI-DP by Dirichlet preconditioner.

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