Optimal Investment, Consumption and Retirement Decision with Disutility and Liquidity Constraints

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ABSTRACT

We consider the general optimization problem in which a working investor has liquidity constraints. We obtain the closed-form solutions for the utility maximization problem by the martingale approaches developed in Karatzas and Wang [14]. The retirement time is the optimal stopping time obtained by solving a variational inequality, which is a free boundary value problem. We follow the approaches of He and Pagès [9] and Dybvig and Liu [7] for the liquidity constraints. The numerical results suggest that the restriction to borrow future income makes the investor retire in a lower critical wealth level than in the case of no liquidity constraints for the case of the constant relative risk aversion(CRRA) utility function.

INTRODUCTION

The optimal consumption and portfolio choice problem has been developed in various strands and under rational constraints. In these days a voluntary retirement has an important role to personal decision in economy. This is why an investor usually works and has disutility which indicates the degree of repugnance about labor. So an investor will retire as soon as her wealth is greater than or equals to a certain critical wealth level. (See [4], [5], [7] and [8].)

The optimal consumption and portfolio decision problem in continuous time was first considered by Merton [16,17]. He introduced the dynamic programming method to solve this problem. This work has been extended to a general utility function by Karatzas et al. [11]. Dynamic programming method has the restriction that price processes are Markov state processes, so dynamic programming method is not allowable for the non-Markov processes. Non-Markov processes cases can be complemented by the martingale method which is derived by Cox and Huang [6] and Karatzas, Lehoczky, and Shreve [12].

Bodie, Merton, and Samuelson [2] considered an optimal consumption and portfolio selection problem of an economic investor who has flexibility in labor supply and showed that flexibility in labor supply tends to increase the agent's risk taking in market securities. Bodie et al. [1] solved an optimal consumption-portfolio selection problem with a fixed time of retirement. The voluntary retirement was resolved by Karatzas and Wang [14]. They used martingale method for discretionary stopping time problem for the first time. Choi and Shim [4] solved a general optimal consumption-portfolio problem with labor income and disutility but they didn't consider the liquidity constraints. (We will consider this model as the *benchmark model*.) Farhi and Panageas [8] considered not only the voluntary retirement but also the liquidity constraints.

Choi, Shim, and Shin [5] solved the problem similar to the one of Farhi and Panageas [8] with the constant elasticity of substitution(CES) utility function.

In this paper, we consider the general optimization problem in which a working investor has liquidity constraints. During the investor works, she receives a labor income and has disutility which indicates a degree of repugnance about working. For computational convenience, only two assets are considered, which are a risky asset and a riskless asset. The assumption makes the value function and the optimal policies be expressed explicitly.

For the case of the general utility function, we obtain the closed-form solutions for the utility maximization problem by the martingale approaches developed in Karatzas and Wang [14]. The retirement time is the optimal stopping time obtained by solving a variational inequality, which is a free boundary value problem. We follow the approaches of He and Pagès [9] and Dybvig and Liu [7] for the liquidity constraints. This procedure is analogous to Farhi and Panageas [8].

The numerical results suggest that the liquidity constraints which prevent borrowing against future income make the investor retire in a lower critical wealth level than in the case of no liquidity constraints. We show the results with the constant relative risk aversion(CRRA) utility function in Section 6. In the examples, it can be also shown that there is a jump in portfolio at the retirement time.

THE FINANCIAL MARKET AND THE OPTIMIZATION PROBLEM

We assume that there are two assets which are one riskless asset with constant interest rate $r > 0$ and one risky asset S_t satisfying the following stochastic differential equation(SDE) $dS_t/S_t = \mu dt + \sigma dB_t,\ S_0 > 0$, where μ and σ are positive constants and B_t is a standard Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\mathcal{F}\}_{t=0}^{\infty}$ is the augmentation of the filtration generated by B_t and $\mathbb{P}$ is the probability measure on $\mathcal{F}$.

We define the market-price-of-risk, the discount process, the exponential martingale and the state-price-density process, respectively, as

$$\theta \triangleq \frac{\mu - r}{\sigma}, \quad \zeta_t \triangleq \exp\{-rt\}, \quad Z_t \triangleq \exp\left\{-\theta B_t - \frac{1}{2}\theta^2 t\right\} \quad \text{and} \quad H_t \triangleq \zeta_t Z_t.$$

We also define an equivalent martingale measure as $\widetilde{\mathbb{P}}(A) \triangleq \mathbb{E}[Z_T \mathbf{1}_A]$, for any given fixed $T > 0$ and any $A \in \mathcal{F}_T$. Then Girsanov theorem gives that the new process $\widetilde{B}_t \triangleq B_t + \theta t$, for $0 \le t \le T$, is a standard Brownian motion under the new measure $\widetilde{\mathbb{P}}$.

Let c_t be an $\mathcal{F}_t$-progressively measurable consumption process $c_t : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ with $\int_0^\infty c_s ds < \infty,$ almost surely(a.s.) and π_t be an $\mathcal{F}_t$-progressively measurable portfolio process $\pi_t : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ with $\int_0^\infty \pi_s^2 ds < \infty,$ a.s.

Let τ be an $\mathcal{F}$-stopping time considered as the retirement time and X_t be the wealth process with the initial endowment $X_0 = x \ge 0$. Then for the given consumption process c_t and portfolio process π_t, X_t evolves

$$dX_t = \left[rX_t + \pi_t(\mu - r) - c_t + \epsilon \mathbf{1}_{\{0 \le t < \tau\}}\right] dt + \pi_t \sigma dB_t \tag{0.1}$$

$$= \left[rX_t - c_t + \epsilon \mathbf{1}_{\{0 \le t < \tau\}}\right] dt + \pi_t \sigma d\widetilde{B}_t, \tag{0.2}$$

where ϵ is a constant labor income before retirement time τ. Then the pair (c, π) is called admissible if $X_t \ge 0$ for all $0 \le t \le \tau$. Since the investor cannot borrow future income the wealth should be always nonnegative, that is,

$$X_t \geq 0, \quad \text{for all } t \geq 0. \tag{0.3}$$

The assumption (0.3) is called liquidity constraint(or borrowing constraint).

From the changed wealth process (0.2), we obtain

$$\zeta_\tau X_\tau - \zeta_t X_t = \int_t^\tau \zeta_s(-c_s + \epsilon)ds + \int_t^\tau \zeta_s \pi_s \sigma d\tilde{B}_s. \tag{0.4}$$

For an admissible pair (c, π) before the retirement time τ, the second term on the right-hand side of (0.4) is a continuous, positive $\mathbb{P}$-local martingale, and hence a supermartingale under the measure $\mathbb{P}$ by Fatou's lemma. Therefore the optional sampling theorem implies

$$H_\tau X_\tau - H_t X_t \leq \mathbb{E}\left[-\int_t^\tau H_s c_s ds + \int_t^\tau H_s \epsilon ds\right], \quad \text{for all } 0 \leq t \leq \tau, \quad \tau \in \mathcal{S},$$

where $\mathcal{S}$ denotes the set of all $\mathcal{F}$-stopping time τ's. If $t = 0$, the budget constraint is given by

$$\mathbb{E}\left[\int_0^\tau H_t c_t dt + H_\tau X_\tau - \int_0^\tau H_t \epsilon dt\right] \leq x. \tag{0.5}$$

Also the liquidity constraint (0.3) implies

$$\mathbb{E}\left[\int_t^\tau H_s c_s ds + H_\tau X_\tau - \int_t^\tau H_s \epsilon ds \middle| \mathcal{F}_t\right] \geq 0, \quad \text{for all } 0 \leq t \leq \tau. \tag{0.6}$$

(See He and Pagès [9] and Farhi and Panageas [8].)

Problem For the given initial endowment x and the utility function $u(c)$, find the value function defined by

$$\begin{aligned} V(x) &= \sup_{(c,\pi,\tau)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left\{u(c_t) - l\mathbf{1}_{\{0\leq t<\tau\}}\right\}dt\right] \\ &= \sup_{(c,\pi,\tau)\in\mathcal{A}} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\{u(c_t) - l\}dt + e^{-\beta\tau}U(X_\tau)\right], \end{aligned} \tag{0.7}$$

subject to the liquidity constraint (0.3) and the budget constraint (0.5).

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