The generalized interface difference method for elliptic problems

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ABSTRACT

In this work, we develop finite difference methods for elliptic problems in the following form

$$-\nabla \cdot (\kappa \nabla u) + c\,u = f$$

in a region Ω in one or two dimensions including an interface Γ, where κ, c and f may have discontinuities across the interface and along the interface f may have a singular source. First, we introduce the generalized Lagrange interpolation formula(GLI) with discontinuities at some points(one dimensional interfaces). Based on this GLI formula, the numerical derivatives up to second order are newly proposed on the uniform grid. We call this the generalized interface difference method(GIDM).

INTRODUCTION

Let us consider the following elliptic equation

$$-\nabla \cdot (\kappa \nabla u) + c\,u = f \tag{1}$$

in a domain Ω with an interface Γ. The solution of or its derivatives might have discontinuities from the presence of discontinuities in the coefficients across the interface or the singular sources along the interface. Therefore, the standard finite different method will not give good approximations for the solutions near the interface. Here we introduce the generalized interpolation of $u(x)$ with piecewise bounded derivatives up to n-th order in an interval $[x_0, x_n]$ with $x_0 < x_1 < \cdots < x_n$ for some positive integer n. As a simplest case, $\alpha \in [x_0, x_n]$(1D interface) is assumed to be the discontinuous point of $u(x)$ or its derivatives and, for example, jump values of its derivatives at α

$$[u]_\alpha,\ [u']_\alpha,\ [u'']_\alpha,\ \cdots,\ [u^{(n)}]_\alpha$$

are known a priori where the bracket notation $[\bullet]_\alpha$ means the difference of $\bullet$ on both sides of the interface α, i.e.,

$$[u]_\alpha = u^+ - u^-$$

where $u^- = \lim_{x\to\alpha-} u(x)$ and $u^+ = \lim_{x\to\alpha+} u(x)$.

Using the well-known piecewise Lagrange interpolation polynomial $L_k(x)$ up to n-th order that passes through n data points

$$(x_0, u(x_0)),\ (x_1, u(x_1)),\ \cdots\ (x_n, u(x_n)), \tag{2}$$

we can find the n-th order generalized Lagrangian interpolation polynomial(GLI formula), $\Phi_n(x)$. For simplicity the second order GLI formula is given at three data points $(x_{i-1}, u(x_{i-1}))$, $(x_i, u(x_i))$ and $(x_{i+1}, u(x_{i+1}))$:

$$\begin{aligned}\Phi_2(x) = S_2(x) &- [\, S_2(x_{i-1})\, L_{i-1}(x;h) + S_2(x_i)\, L_i(x;h) + S_2(x_{i+1})\, L_{i+1}(x;h)\,] \\ &+ [\, u(x_{i-1})\, L_{i-1}(x;h) + u(x_i)\, L_i(x;h) + u(x_{i+1})\, L_{i+1}(x;h)\,]\,,\end{aligned} \tag{3}$$

where we define

$$S_2(x) = \sum_{j=1}^{m} \underbrace{[u]_{\alpha_j}\, H(x-\alpha_j)}_{Step} + \underbrace{\frac{1}{2}[u']_{\alpha_j}|x-\alpha_j|}_{First\ Wedge} + \underbrace{\frac{1}{4}[u'']_{\alpha_j}|x-\alpha_j|(x-\alpha_j)}_{Second\ Wedge}, \tag{4}$$

and

$$L_{i-1}(x;h) = \frac{(x-x_i)(x-x_{i+1})}{2h^2} \tag{5}$$

$$L_i(x;h) = -\frac{(x-x_{i-1})(x-x_{i+1})}{h^2} \tag{6}$$

$$L_{i+1}(x;h) = \frac{(x-x_{i-1})(x-x_i)}{2h^2} \tag{7}$$

where $H(t)$ is a Heaviside function, in our case,

$$H(t) = \begin{cases} -\frac{1}{2} & t < 0, \\ \frac{1}{2} & t \geq 0. \end{cases} \tag{8}$$

Now having the second order GLI formula at hand it is possible to maintain second order accurate approximations near the interface for $u(x)$ which has discontinuities at α_j, $j = 1, \cdots, m,$. At a node point x_i with next nearest nodes x_{i-1} and x_{i+1}, the approximations of derivatives up to second order, $u'_h(x_i)$ and $u''_h(x_i)$ respectively for $u'(x_i)$ and $u''(x_i)$ are defined as follows

$$u'_h(x_i) = \Phi'_2(x_i), \tag{9}$$

$$u''_h(x_i) = \Phi''_2(x_i) \tag{10}$$

where $\Phi_2(x)$ is in (3) with three nodes, x_{i-1}, x_i, and x_{i+1}, in which

$$x_{i-1} = x_i - h, \quad x_{i+1} = x_i + h \quad (h > 0). \tag{11}$$

Since a priori information to complete the GLI function are the discrete values $u(x_i)$ as well as the jump values on the interface α_j, $[u]_{\alpha_j}$, $[u']_{\alpha_j}$, and $[u'']_{\alpha_j}$, an explicit form of approximations which reveals the unknowns is more useful to solve problems under consideration. Further simplifying of (9) and (10) at the node point x_i are written as

$$u_h'(x_i) = \frac{u(x_{i+1}) - u(x_{i-1})}{2h} + \underbrace{\sum_{j=1}^{m} [u]_{\alpha_j} \gamma_i^{(0)}(\alpha_j) + [u']_{\alpha_j}\, \gamma_i^{(1)}(\alpha_j) + [u'']_{\alpha_j}\, \gamma_i^{(2)}(\alpha_j)}_{Singular\ Approximation} \tag{12}$$

$$u_h''(x_i) = \frac{u(x_{i-1}) - 2\,u(x_i) + u(x_{i+1})}{h^2} + \underbrace{\sum_{j=1}^{m} [u]_{\alpha_j}\, \bar{\gamma}_i^{(0)}(\alpha_j) + [u']_{\alpha_j}\, \bar{\gamma}_i^{(1)}(\alpha_j) + [u'']_{\alpha_j}\, \bar{\gamma}_i^{(2)}(\alpha_j)}_{Singular\ Approximation} \tag{13}$$

We call this approximation method the generalized interface difference method or briefly the GIDM. As a matter of fact, if $\alpha_j \notin (x_{i-1}, x_{i+1})$ for all $j = 1, \cdots, m$, then both approximations in (12) and (13) equal to the standard finite difference because the singular parts of both approaimations vanish.

NUMERICAL EXAMPLE

We have done some numerical examples which validate the second order accuracy of the GIDM method. We present the following problem:

$$-(\kappa\, u')' = 12x^2, \quad \text{in } (0,1) \setminus \{\alpha\}, \tag{14}$$

$$u(0) = 0, u(1) = \frac{1}{\kappa^+} + (\frac{1}{\kappa^-} - \frac{1}{\kappa^+})\alpha^4, \tag{15}$$

$$[u]_\alpha = 0, [\kappa\, u']_\alpha = 0 \tag{16}$$

where $\alpha \in (0,1)$ is an interface, κ is κ^- when $x < \alpha$ and κ^+ when $x > \alpha$.

The exact solution is

$$u(x) = \begin{cases} -\frac{x^4}{\kappa^-} & x < \alpha, \\ \frac{x^4}{\kappa^+} + (\frac{1}{\kappa^-} - \frac{1}{\kappa^+})\alpha^4 & x > \alpha. \end{cases} \tag{17}$$

Figures 1 and 2 show comparisons of numerical solutions with exact solutions with two different κ ratios. Both solutions are in a very good agreement with exact solutions. Figure 3 gives the maximum error and slope 2 corresponds to the second order accuracy for the solution u and slope 1 first order accuracy for its derivative u'.

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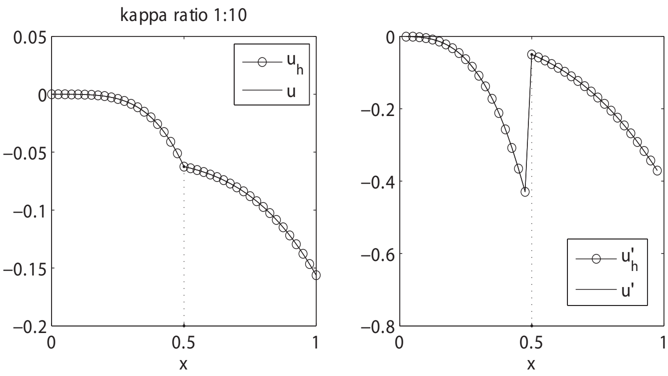

Figure 1. Comparison of the GIDM solution with the exact solution for κ ratio 1 to 10.

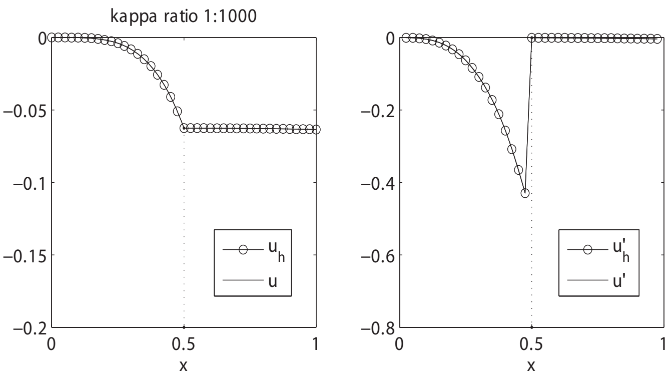

Figure 2. Comparison of the GIDM solution with the exact solution for κ ratio 1 to 1000.

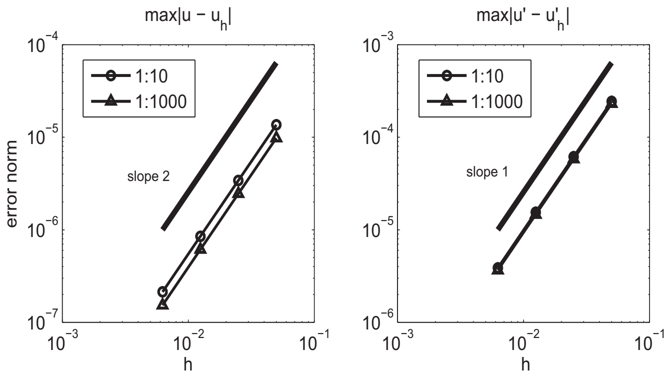

Figure 3. Maximum error of solution and its derivative.

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