ALIASING ERROR OF SAMPLING SERIES IN WAVELET SUBSPACES

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ABSTRACT

In [6], Walter extended the Shannon sampling theorem [3], under appropriate hypotheses, the Shannon sampling theorem to the subspace V_0 of a general multiresolution analysis $\{V_n\}_{n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$: For any $f \in V_0$ the sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n)S(t-n), \quad t \in \mathbb{R}$$

holds, where $\widehat{S}(\xi) := \widehat{\phi}(\xi)/(\sum_{n\in\mathbb{Z}} \phi(n)e^{-in\xi})$ and ϕ denotes the scaling function. Later on, Unser and Aldroubi introduced in [5], under suitable conditions, the average sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} (\mathcal{L}f)(n)\; S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,,$$

which uses the average samples $\{(\mathcal{L}f)(n)\}_{n\in\mathbb{Z}}$ obtained from $f \in V_0$ by means of a linear time-invariant system $(\mathcal{L}f) = f * \ell$ defined on V_0.

Whenever these sampling formulas are applied to a function f which does not belong to V_0, the so-called aliasing error arises;

$$E^A f(t) := f(t) - \sum_{n\in\mathbb{Z}} f(n)S(t-n) \;\text{ or }\; E^A_{\mathcal{L}} f(t) := f(t) - \sum_{n\in\mathbb{Z}} (\mathcal{L}f)(n)S_{\mathcal{L}}(t-n)\,, \quad t \in \mathbb{R}\,.$$

Concerning this error in Shannon's setting, a classic result by Brown [1] states that if $f \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then

$$\Big| f(t) - \sum_{n\in\mathbb{Z}} f(n)\mathrm{sinc}(t-n) \Big| \le \frac{2}{\sqrt{2\pi}} \int_{|\xi|>\pi} |\widehat{f}(\xi)| d\xi\,, \quad t \in \mathbb{R}\,. \tag{1}$$

Notice that if $f \in V_1 = PW_{2\pi}$, then (1) can be written as

$$|E^A f(t)| \le \frac{2}{\sqrt{2\pi}} \|P_{W_0} f\|_{L^1(\mathbb{R})}\,,$$

where P_{W_0} denotes the orthogonal projection onto W_0, the orthogonal complement of V_0 in V_1. The aliasing error in Shannon's setting has been largely studied: see [3] and references therein.

Besides, Walter [6] has proved a similar result for functions in the subspace V_1 of a general multiresolution analysis. Specifically, for any $f \in V_1$, there exists a constant C such that

$$|E^A f(t)| \leq C \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}\,. \tag{2}$$

On the other hand, Janssen generalized in [4] Walter's sampling formula by using shifted samples $\{f(n+\sigma)\}_{n\in\mathbb{Z}}$, where $\sigma \in [0,1)$. As to the corresponding aliasing error $E^A f$, he proved the inequalities

$$K_0 \|P_{W_0} f\|_{L^2(\mathbb{R})} \leq \|E^A f\|_{L^2(\mathbb{R})} \leq K_\infty \|P_{W_0} f\|_{L^2(\mathbb{R})}, \quad f \in V_1\,.$$

In addition, he found the smallest possible value for the constant K_0 and the largest possible value for K_∞. Later on, in [2] the authors dealt with the aliasing error function $E^A f$ for $f \in V_1$. In so doing, they calculate its Fourier transform, $\widehat{E^A f}$, in terms of the Fourier transform of $P_{W_0} f$. Besides recovering Janssen's inequalities, this technique also allows to derive a precise bound like (2), exhibiting the extremal solutions in some cases. Some results concerning the aliasing error for functions $f \in V_2$ are also provided. See also references [5] and [7] for the general wavelet setting.

In this talk, we discuss the aliasing error arising when the classical sampling formula is applied to a function f in the wavelet subspace V_n, $n \geq 1$, of a multiresolution analysis. Estimations both in L^2 and L^∞ norms are provided. The alising error arising when we apply the average sampling formula for V_0 to a function $f \in V_1$ is also included. In particular, this paper improves the results in [2] in different directions: Apart from to work with a non necessarily orthonormal scaling function ϕ, some of the results in [2] are derived under weaker hypotheses.

In this talk, we consider the aliasing error in classical sampling for wavelet subspaces in a multiresolution analysis. Also we consider the aliasing error in average sampling.

REFERENCES

1. J. L. Brown, Jr., "On the error in reconstructing non-bandlimited function by means of the bandpass sampling theorem", *J. Math. Anal. Appl.*, Vol. 18, 1967, pp. 75–84.
2. A. G. García and G. Pérez-Villalón, "On the aliasing error in wavelet subspaces", *J. Comput. Appl. Math.*, Vol. 183, 2005, pp. 153-167.
3. J. R. Higgins, *Sampling theory in Fourier and signal analysis : Foundations*, Oxford Univ. Press, Oxford, 1996.
4. A. J. E. M. Janssen, "The Zak transform and sampling theorems for wavelet subspaces", *IEEE Trans. Signal Processing*, Vol. 41, 1993, pp. 3360-3364.
5. M. Unser and A. Aldroubi, "A general sampling theory for nonideal acquisition devices", *IEEE Trans. on Signal Process.*, Vol. 42, no. 11, 1994, pp. 2915-2925.
6. G. G. Walter, "A sampling theorem for wavelet subspaces", *IEEE Trans. Inform. Theory*, Vol. 38, 1992, pp. 881-884.
7. X. G. Xia and Z. Zhang, "Estimation of aliasing error in sampling theorem for signals not necessarily in wavelet subspaces", *Proc. of ICASSP-93*, Vol. 3, pp 300-303, 1993.