CONSTRUCTION OF A NON-STATIONARY BIORTHOGONAL WAVELET SYSTEM USING A SUBDIVISION SCHEME

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ABSTRACT

We present the 4-point Gaussian based non-stationary interpolatory subdivision scheme which approximates the given data in the span of translations for the Gaussian function with the parameter λ. Different choice for the parameter leads to a different set of masks. The sequence of the refinable functions converges to the refinable function for the 4-point Deslauriers-Dubuc scheme in $L^p(0 < p \leq \infty)$. Using the subdivision mask as the lowpass filter, we construct the non-stationary biorthogonal Multiresolution Analysis(MRA) which has the approximation order of 4. Then we derive the corresponding non-stationary biorthogonal wavelet system. The resulting scaling functions and the wavelets are symmetric and compactly supported.

INTRODUCTION

Subdivision and wavelets can be triggered by the **two-scaling relation**(or **refinement equation**)

$$\phi(x) = \sum_{n \in \mathbf{Z}} a_n \phi(2x - n). \tag{1}$$

In this talk, we employ the Gaussian based interpolatory subdivision scheme. The scheme is **non-stationary**, which means that we have a sequence of masks that varies along the subdivision level(k). In this case, two-scaling relation evolves into the next form;

$$\phi^{[k]}(x) = \sum_{n \in \mathbf{Z}} a_n^{[k+1]} \phi^{[k+1]}(2x - n). \tag{2}$$

Hence the wavelet system becomes non-stationary (Figure 1).

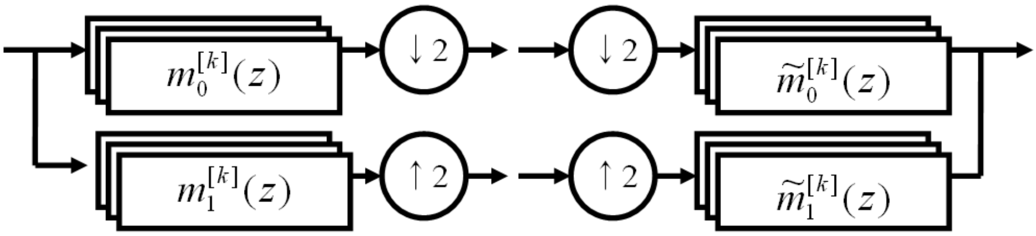


Figure 1. Filter bank for a non-stationary biorthogonal wavelet system

$level(k)$	$a_{-3}^{[k]} = a_3^{[k]}$	$a_{-1}^{[k]} = a_1^{[k]}$
$k=0$	0.14533497825242087	0.35466502174757913
$k=1$	−0.0827693809671577	0.5827693809671577
$k=2$	−0.10336472362781433	0.6033647236278145
$k=3$	−0.07393275304573729	0.57393
$k=4$	−0.06536522668923167	0.5653
$k=5$	−0.06321559435474657	0.56321
$k=6$	−0.0626788358608841	0.56267
$k=7$	−0.06254458154304064	0.56254
$k=8$	−0.06251115957350352	0.56251
DD	−0.06250000000000000	0.5625

$\lambda = 0.4$

$\lambda = 1.8$

$level(k)$	$a_{-3}^{[k]} = a_3^{[k]}$	$a_{-1}^{[k]} = a_1^{[k]}$
$k=0$	−0.09636438282336211	0.5963643828233622
$k=1$	−0.0715489455090271	0.5715489455090387
$k=2$	−0.06476339837689195	0.5647633983774237
$k=3$	−0.06306535499417792	0.563065354904183
$k=4$	−0.06264129887896984	0.5626412984713349
$k=5$	−0.06253506771865427	0.5625345594519919
$k=6$	−0.06251040909387304	0.5625135652619128
$k=7$	−0.06281450265323035	0.5634390476098402
$k=8$	−0.06250074226276389	0.5625007456376836
DD	−0.06250000000000000	0.5625000000000000

Figure 2. Subdivision masks for $\lambda = 0.4$ and $\lambda = 1.8$. Since the scheme is interpolatory, $a_{2n}^{[k]} = \delta_{0,n}$. Lowpass filters are $h_n^{[k]} := \frac{1}{\sqrt{2}} a_n^{[k]}$.

ALGORITHMS

From the initial data $X = \{(x_n, f(x_n)) | n = -m+1, ..., m\}$, where $f : \mathbf{R} \to \mathbf{R}$, we need to find an approximation $\mathcal{I}_{f,X}$ from the space $span\{G(\cdot - x_{-m+1}), ..., G(\cdot - x_m)\} := G_X$, where $G(x) = e^{-\frac{x^2}{\lambda^2}}$.

$$\mathcal{I}_{f,X}(x) = \sum_{n=-m+1}^{m} S_n(x) f(x_n) \tag{3}$$

where $S_n(x_l) = \delta_{n,l}$ and $span\{S_1(x), ..., S_N(x)\} = G_X$. Different choice for the parameter leads to a different set of masks. The mask converges to the mask for the Deslauriers-Dubuc scheme (Figure 2) for any choice of λ.

Using the subdivision mask, we can define the corresponding lowpass filter.

$$m_0^{[k]}(\xi) := \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} h_n^{[k]} e^{in\xi} \equiv \frac{1}{2} \sum_{n \in \mathbf{Z}} a_n^{[k]} e^{in\xi} \tag{4}$$

To construct a non-stationary biorthogonal MRA, we are looking for a symmetric dual filter satisfying

$$m_0^{[k]}(\xi)\overline{\tilde{m}_0^{[k]}(\xi)} + m_0^{[k]}(\xi+\pi)\overline{\tilde{m}_0^{[k]}(\xi+\pi)} = 1, \qquad {}^\forall k \in \mathbf{Z} \tag{5}$$

where and $\widetilde{m}_0^{[k]}(\xi) := \frac{1}{\sqrt{2}} \sum_{n \in \mathbf{Z}} \widetilde{h}_n^{[k]} e^{in\xi}$. To determine such a $\widetilde{m}_0^{[k]}(\xi)$, take

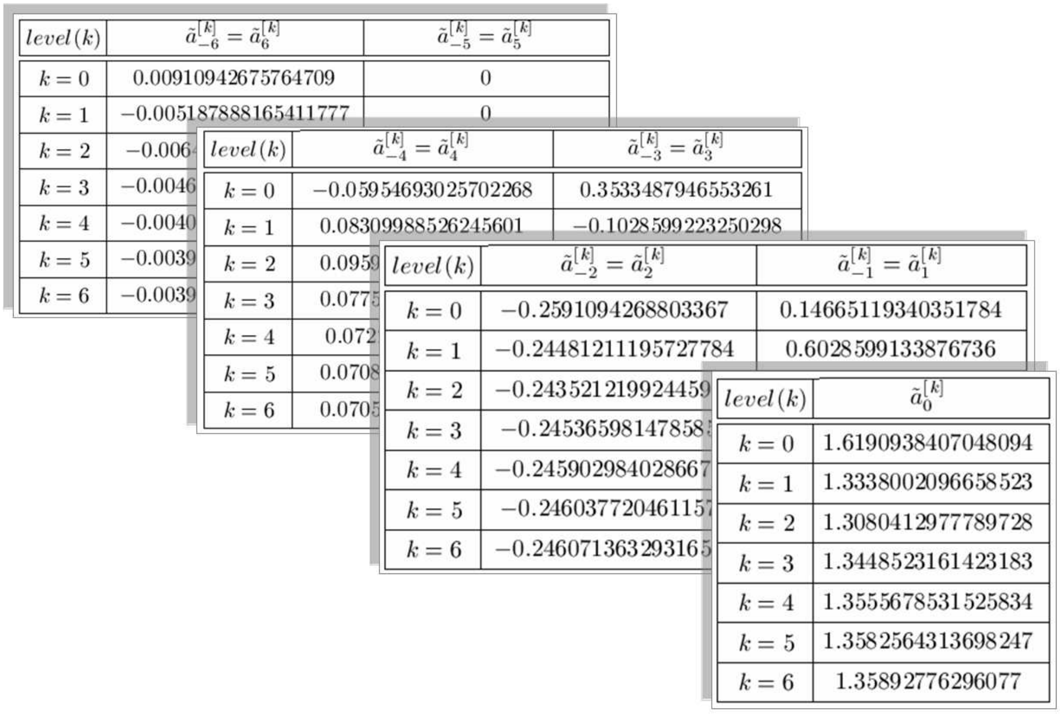
$$\tilde{m}_0^{[k]}(\xi) := m_0^{[k]}(\xi) q^{[k]}(y), \tag{6}$$

where $y = \sin^2 \frac{\xi}{2}$ and $q^{[k]}$ is a polynomial, and then apply the *Bezout theorem* (Figure 3).

Being an approximation to the original function $f \in L^2(\mathbf{R})$, the biorthogonal projection on each level,

$$P^{[k]} f(x) := \sum < f(\cdot), \tilde{\phi}^{[k]}(2^k \cdot - n) > \phi^{[k]}(2^k x - n) \tag{7}$$

converges to the original function f as $k \to \infty$. The approximation order estimates how fast the error (in L^2 norm) decays as the level k proceeds.



$level(k)$	$\tilde{a}_{-6}^{[k]} = \tilde{a}_{6}^{[k]}$	$\tilde{a}_{-5}^{[k]} = \tilde{a}_{5}^{[k]}$
$k=0$	0.00910942675764709	0
$k=1$	−0.005187888165411777	0
$k=2$	−0.006	
$k=3$	−0.0046	
$k=4$	−0.0040	
$k=5$	−0.0039	
$k=6$	−0.0039	

$level(k)$	$\tilde{a}_{-4}^{[k]} = \tilde{a}_{4}^{[k]}$	$\tilde{a}_{-3}^{[k]} = \tilde{a}_{3}^{[k]}$
$k=0$	−0.05954693025702268	0.3533487946553261
$k=1$	0.08309988526245601	−0.1028599223250298
$k=2$	0.0959	
$k=3$	0.0775	
$k=4$	0.072	
$k=5$	0.0708	
$k=6$	0.0705	

$level(k)$	$\tilde{a}_{-2}^{[k]} = \tilde{a}_{2}^{[k]}$	$\tilde{a}_{-1}^{[k]} = \tilde{a}_{1}^{[k]}$
$k=0$	−0.2591094268803367	0.14665119340351784
$k=1$	−0.24481211195727784	0.6028599133876736
$k=2$	−0.243521219924459	
$k=3$	−0.24536598147858	
$k=4$	−0.245902984028667	
$k=5$	−0.24603772046115	
$k=6$	−0.24607136329316	

$level(k)$	$\tilde{a}_{0}^{[k]}$
$k=0$	1.6190938407048094
$k=1$	1.3338002096658523
$k=2$	1.3080412977789728
$k=3$	1.3448523161423183
$k=4$	1.3555678531525834
$k=5$	1.3582564313698247
$k=6$	1.35892776296077

Figure 3. Symmetric dual filter $\tilde{h}_n^{[k]} := \frac{1}{\sqrt{2}}\tilde{a}_n^{[k]}$ for $\lambda = 0.4$.

Using the following two facts;
(1) the biorthogonal MRA with ϕ_D (the basic limit function for the Deslauriers-Dubuc subdivision scheme) reproduces polynomials with degree up to 4,
(2) $||\phi^{[k]} - \phi_D||_\infty < c\frac{1}{2^{2k}}$,
we can conclude that our non-stationary biorthogonal MRA has the approximation order 4. That is,

$$||f - P^{[k]}f||_2 < C\frac{1}{2^{4k}}, \quad \forall k \in \mathbf{Z}. \tag{8}$$

Having the $m_0^{[k]}$ and $\tilde{m}_0^{[k]}$ at hand, an easy choice for $m_1^{[k]}$ and $\tilde{m}_1^{[k]}$ satisfying

$$\begin{pmatrix} m_0^{[k]}(\xi) & m_0^{[k]}(\xi+\pi) \\ m_1^{[k]}(\xi) & m_1^{[k]}(\xi+\pi) \end{pmatrix} \begin{pmatrix} \overline{\widetilde{m}_0^{[k]}(\xi)} & \overline{\widetilde{m}_1^{[k]}(\xi)} \\ \overline{\widetilde{m}_0^{[k]}(\xi+\pi)} & \overline{\widetilde{m}_1^{[k]}(\xi+\pi)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \forall k \in \mathbf{Z}. \tag{9}$$

would be

$$m_1^{[k]}(\xi) := e^{-i\xi}\overline{\widetilde{m}_0^{[k]}(\xi+\pi)} \quad and \quad \widetilde{m}_1^{[k]}(\xi) := e^{-i\xi}\overline{m_0^{[k]}(\xi+\pi)}. \tag{10}$$

Figure 4 shows the shape of wavelet systems varying the value of λ. For the convenience of the comparision, we fixed the level $k = 0$.

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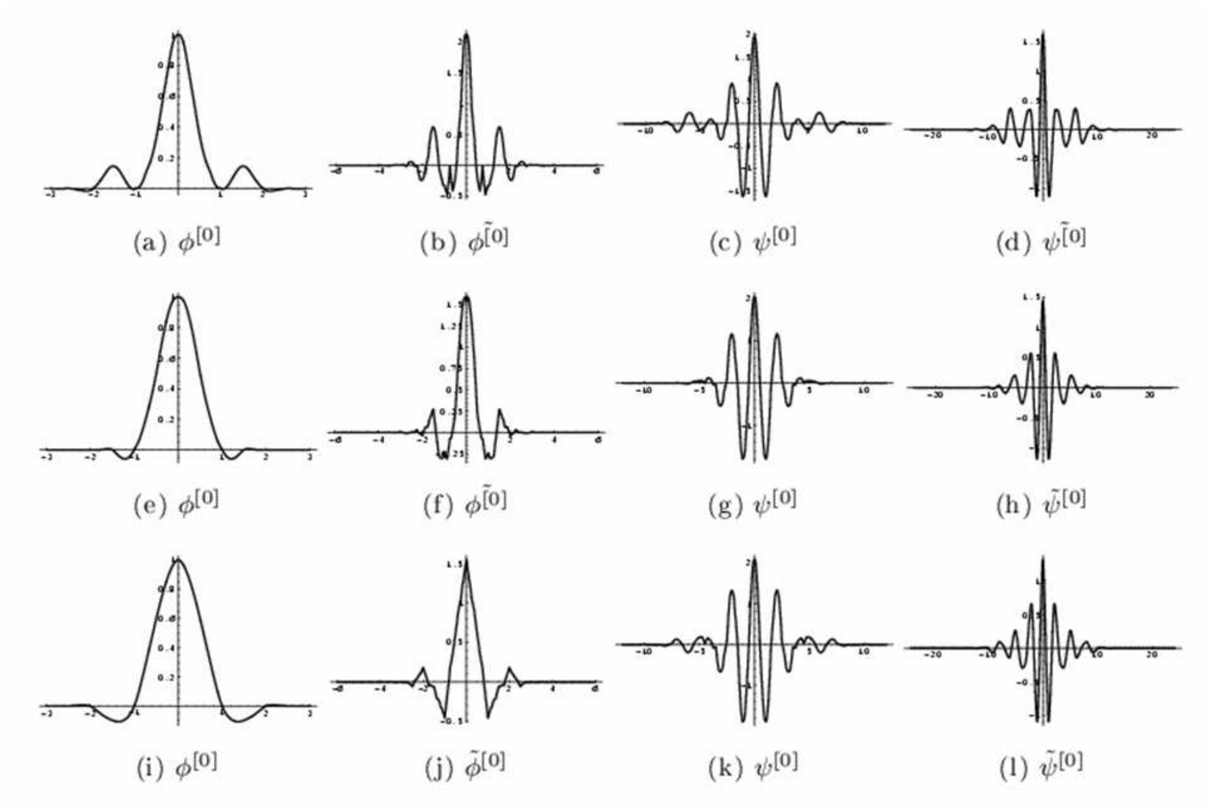

Figure 4. Wavelet systems based on the Gaussian function, $G(x) = e^{-\frac{x^2}{\lambda^2}}$. From the top, each row corresponds to $\lambda = 0.4$, $\lambda = 0.6$, and $\lambda = 1.8$. From the left, each column corresponds to the scaling functions $\phi^{[0]}, \tilde{\phi}^{[0]}$, and the wavelet functions $\psi^{[0]}, \tilde{\psi}^{[0]}$ respectively.

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