Mutual Capacitance via Last-passage Algoritms

Chi-Ok Hwang[1] and James A. Given[2]

1) *National Institute for Mathematcial Sciences, Daejeon, KOREA*

2) *Naval Research Lab, Washing, USA*

Corresponding Author : Chi-Ok Hwang, e-mail@nims.re.kr

ABSTRACT

We develop and test the last-passage diffusion algorithm for the mutual capacitance of a system of conductors. This is the first charge-based Monte Carlo algorithm for this quantity. The first-passage algorithm is highly efficient because it is charge-based, and incorporates importance sampling; it averages over the properties of Brownian paths that initiate outside the conductor and terminate on its surface. However, this algorithm does not seem to generalize to mutual capacitance problems. The last passage algorithm, in a sense, is the time reversal of the first-passage algorithm; it involves averages over particles that initiate on an absorbing surface, leave that surface, and diffuse away to infinity. To validate this algorithm, we calculate the mutual capacitance matrix of the circular disk parallel plate capacitor and compare with the known numerical results. Good agreement is obtained.

1 FORMALISM AND A SIMPLE EXAMPLE

In this talk, we introduce the last-passage diffusion algorithm [10,13], a Monte Carlo method, which does allow calculation of mutual capacitance. In this method, we average over diffusion paths that start very near to one of the absorbing objects and diffuse until they are either absorbed by a different absorbing object, or diffuse away to infinity.

The mutual capacitance matrix C_{ji} of a set of N conductors is defined by the relation:

$$Q_j = \sum_{i=1}^{N} C_{ji} V_i \tag{1}$$

Here, V_j and Q_j are respectively the voltage and charge on the j^{TH} conductor. C_{ji} is the total charge on conductor j, when one applies unit voltage to conductor i while grounding all the other conductors. The voltage field $V(x)$ thus imposed is identical to the function $P(x \rightarrow C^i)$, which is the probability that a diffusing particle started at the point x will be absorbed on the surface C^i of the i^{TH} conductor. By Gauss' law, the surface charge density $\sigma_{ii}(x)$ is given by

$$\sigma_{ii}(x) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} P\left[(x+\epsilon) \rightarrow C^i\right] = \lim_{\epsilon \rightarrow 0} \frac{P\left[(x+\epsilon) \rightarrow C^i\right] - 1}{\epsilon} \tag{2}$$

Except for a minus sign, the numerator of this expression is the probability that a diffusing particle starting at $(x+\epsilon)$ will **not** be absorbed on the i^{TH} surface. The $\epsilon \rightarrow 0$ limit is evaluated

as follows: the probability $P\left[(x+\epsilon) \rightarrow C^i\right]$ can be written as a convolution of two factors; the probability $g(x+\epsilon, y)$ that a diffusing particle leave point $(x+\epsilon)$ and make first passage at a point y on a sphere of radius a surrounding point x; and the probability $\left[1 - P(y \rightarrow C^i)\right]$ that a particle started at the point y be absorbed by a conductor other than the i^{TH}, or wander off to infinity. The first factor is proportional to ϵ, but is simple. The second factor can be obtained via Monte Carlo simulation. Thus C_{ii} is given by

$$C_{ii} = \oint_{x \in C^i} dS \sigma_{ii}(x) \tag{3}$$

$$\sigma_{ii}(x) = \frac{1}{4\pi} \oint d^2y G(x,y) \left[1 - P(y \rightarrow C^i)\right] \tag{4}$$

Here, $G(x,y)$ is the Laplacian Green's function for a point dipole centered at point x and normal to the surface;

$$G(\mathbf{x},\mathbf{y}) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g(\mathbf{x}+\epsilon,\mathbf{y}) \tag{5}$$

where $g(\mathbf{x}+\epsilon,\mathbf{y})$ is the Laplacian Green's function associated with Dirichlet boundary conditions on the region $\partial\Omega_y$. For a flat conducting surface, this dipole Green's function is given by [10]

$$G(\mathbf{x},\mathbf{y}) = \frac{3}{2\pi} \frac{\cos\theta}{a^3}, \tag{6}$$

where θ is the angle between the vectors x and y.

The same ideas give for C_{ij}, $j \neq i$

$$C_{ij} = \oint_{x \in C^i} dS \sigma_{ij}(x) \tag{7}$$

$$\sigma_{ij}(x) = -\frac{1}{4\pi} \oint d^2y G(x,y) P(y \rightarrow C^j). \tag{8}$$

The last-passage method is tested by calculating the mutual capacitance of the circular disk parallel plate capacitor [16]. The integrals over surface charge in Eqns. (3,7) are calculated using the fractional sampling method, [19], with the reference for importance sampling of the charge distribution being the single charged circular plate. The fractional sampling method has been used extensively in neutron transport and similar problems.

We compare our results with those obtained by numerical solution of an integral equation of Fredholm type for the charge density in this problem [16]. There are two independent components of the matrix C_{ij} in this problem, $(C_{11} - C_{12})$ and $(C_{11} + C_{12})$. Carlson and Illman used a numerical approach to calculate the former; we extended this approach to calculate the latter. We obtained good agreement for both.

REFERENCES

1. K.L. Chung, **Green, Brown and Probability**, (World Scientific, N.Y., 1996).
2. C.O. Hwang, J.A. Given, and M. Mascagni, Physics of Fluids **12**, 1699 (2000).
3. H. X. Zhou, A. Szabo, J. F. Douglas and J. B. Hubbard, J. Chem. Phys. **100**, 3821 (1994).

4. S. Torquato, I. C. Kim and D. Cule, J. Applied Phys. **85**, 1560, (1999).
5. K.K. Sabelfeld, **Monte Carlo Methods in Boundary Value Problems**, (Springer, N.Y., 1991).
6. M. E. Muller, Ann Math Stat **27**, 569 (1956).
7. S. A. Allison, S. H. Northrup, and J. A. McCammon, J. Chem. Phys. **83**, 2894 (1985).
8. A. H. Zemanian, R. P. Tewardson, C. P. Ju and J. F. Jen, IEEE Trans Comput. Aided Design. **8**, 1319 (1989).
9. Y. L. Le Coz and R. B. Iverson, Solid-St. Electron. **35**, 1005 (1992).
10. J. A. Given, C.-O. Hwang and M. Mascagni, Phys. Rev. E **66**, 056704 (2002).
11. J. Ambrosiano, L. Greengard, and V. Rokhlin, Computer Phys Comm **48**, 117 (1988).
12. M. L. Mansfield, J. F. Douglas, and E. J. Garboczi, Phys. Rev. E **64(6)**, 061401:1 (2001).
13. J. A. Given and C.-O. Hwang, Phys. Rev. E **68**, 046128:1, (2003).
14. C-O. Hwang and M. Mascagni, J. Appl. Phys. **95(7)**, 3798 (2004).
15. C-O. Hwang and T. Won, J. Korean Phys. Soc. **45**, S551, (2004).
16. G. T. Carlson and B. L. Illman, Am. J. Phys. **62(12)**, 1099 (1994).
17. T.E. Booth, Nuclear Science and Engineering **138**, 96 (2001).
18. J. M. DeLaurentis and L. A. Romero, J. Comput. Phys. **90**, 123, (1990).
19. J. M. Hammetsley and D. C. Handscomb, **Monte Carlo Methods** (Methuen & Co., London, 1964).