THE LINEAR INDEPENDENCE CONJECTURE FOR TIME-FREQUENCY SHIFTS

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ABSTRACT

In this talk, we recall concepts of Schauder and Riesz bases, and frame, which are essential tools in the study of wavelet and time-frequency analysis. Then we introduce an open conjecture in time-frequency analysis on the linear independence of a finite set of time-frequency shifts of a given L^2 function. Despite the striking simplicity of the statement of this conjecture, it remains open today in the generality. The partial results that are known to hold for the conjecture are also presented and discussed.

INTRODUCTION

Frames are an essential tool for many emerging applications such as data transmission. Their main advantages is the fact that frames can be designed to be redundant while still providing reconstruction formulas. This makes them robust against noise and losses while allowing freedom in design.

In the Hilbert space setting, frames are a class of sequences which not only span the whole space but also provide stable reconstruction formulas. Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series as an alternative to orthonormal or Riesz bases in Hilbert spaces.

Definition 1 *A sequence $\{f_n\}$ in a Hilbert space H is a frame for H if there exist $A, B > 0$ (lower and upper frame bounds) such that*

$$A||f||^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B||f||^2, \quad \forall f \in H. \tag{1}$$

Gabor frames are a particular type of frame whose elements are simply generated by time-frequency shifts of a single *window function*. Specifically, if $g \in L^2(\mathbb{R})$ is a fixed function and Λ is a sequence of points in $\mathbb{R}^2$, then the *Gabor system* or *Weyl-Heisenberg system* generated by g and Λ is the set of time-frequency shifts of g along Λ given by

$$\mathcal{G}(g, \Lambda) = \{M_b T_a g\}_{(a,b)\in\Lambda} = \{e^{2\pi ibt} g(t-a)\}_{(a,b)\in\Lambda}. \tag{2}$$

Here T_a is the *translation operator* $T_a g(t) = g(t-a)$ and M_b is the *modulation operator* $M_b g(t) = e^{2\pi ibt} g(t)$. The compositions $T_a M_b$ or $M_b T_a$ are *time-frequency shifts operators*. If the Gabor system is a frame then we call it a Gabor frame. The structure of Gabor frames makes them especially suitable for applications in time-dependent frequency content.

Applying Gabor frame expansions to derive boundedness and spectral results for pseudodifferential operators, Heil, Ramanathan, and Topiwala explored the basic structure of Gabor frames. And they made the following conjecture (sometimes called the HRT conjecture, the Linear Independence Conjecture for Time-frequency Shifts, or the Zero Divisor Conjecture for the Heisenberg Group) ([2]).

Conjecture 1 *If $g \in L^2(\mathbb{R})$ is nonzero and $\{\alpha_k, \beta_k\}_{k=1}^N$ is any set of finitely many distinct points in $\mathbb{R}^2$, then $\{e^{2\pi i \beta_k t} g(t-\alpha_k)\}_{k=1}^N$ is a linearly independent set of functions in $L^2(\mathbb{R})$.*

Despite the striking simplicity of the statement of the conjecture, it remains open today and some partial results were obtained [2] as follows :

(a) If a nonzero $g \in L^2(\mathbb{R})$ is compactly supported, or just supported on a half-line, then the independence conclusion holds for any value N.
(b) If $g(t) = p(t)e^{-t^2}$ where p is a nonzero polynomial, then the independence conclusion holds for any value N.
(c) The independence conclusion holds for any nonzero $g \in L^2(\mathbb{R})$ if $N \leq 3$.
(d) If the independence conclusion holds for a particular nonzero $g \in L^2(\mathbb{R})$ and a particular choice of points $\{\alpha_k, \beta_k\}_{k=1}^N$, then there exists an ε such that it also holds for any h satisfying $||g-h||_2 < \varepsilon$, using the same set of points.
(e) If the independence conclusion holds for a particular nonzero $g \in L^2(\mathbb{R})$ and a particular choice of points $\{\alpha_k, \beta_k\}_{k=1}^N$, then there exists an ε such that it also holds for the g and any set of N points within ε of the original ones.

It is perhaps surprising that there are almost no partial results formulated in terms of smoothness or decay condition on g. In particular, Conjecture 1 is open if we impose the extra hypothesis that g lies in the Schwartz class.

In this talk, we recall concepts of Schauder and Riesz bases, and frame, which are essential tools in the study of wavelet and time-frequency analysis. Then we will discuss why the conjecture is difficult to solve.

REFERENCES

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