GENERALIZED FUNCTION THEORY AND ITS APPLICATIONS

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ABSTRACT

A generalized singularity description of an inhomogeneous wave field for moving surface boundary conditions in arbitrary motion is presented and applied to nonlinear wave propagation and acoustics of rotors in high speed. A nonlinear transonic aerodynamic equation is derived for understanding basic impulsive noise generating mechanisms of rotors in high speed. This equation includes the high-order mathematical singularities in the nonlinear transonic equation in a rotating frame, which is an integral equation with no simple closed-form solution available. The governing equation for high-speed impulsive rotor noise is formulated with a nonlinear velocity potential and this equation governs how local shock waves propagate away from rotating blades. This equation is also useful to explain the phenomenon of delocalization, which is a sudden change of propagation of nonlinear transonic shock waves on a blade to a far-field at a certain Mach number. This delocalization phenomenon is validated with experimental results and explains one of the unsolved acoustic propagation problems. The important aspect of this phenomenon is an introduction of a nonlinear sonic curve concept with respect to nonlinear wave propagation. The effect of the delocalization Mach number on the nonlinear wave propagation of high-speed impulsive rotor noise is investigated.

INTRODUCTION

Over the years, many researchers have investigated the basic noise generating mechanisms of rotating blades in a transonic region. Among several different noise mechanisms, there is one important type of noise, called high-speed impulsive rotor noise. This type of noise is very important due to a fact that this can be used for acoustic detection in military applications and is also responsible for noise pollution to friendly-neighbors.

This type of noise has been investigated analytically and experimentally by many researchers in the past. Experimentally, many experiments have been carried out with flight tests and wind tunnel tests, including a hover chamber to investigate the noise generation mechanisms and the noise propagation to a far-field [1-2]. One of many interesting experimental results was the sudden waveform changes from smooth waveform to shock waveform with a small change of rotational speed, called the “delocalization” phenomenon. The delocalization phenomenon is a newly observed phenomenon and directly related to the nonlinear propagation of shock wave to a far-field, and this is entirely dependent on local flow field on the blade and the nonlinear sonic curves [2]. The tip Mach number of a blade is called “delocalization Mach number” when the delocalization phenomenon is initiated and the concept of a delocalization Mach number is very useful to design a rotor blade for less noise.

Analytically, the aerodynamically-generated noise can be modeled by using the Lighthill acoustic analogy [3]. The Lighthill analogy is a very useful linearization of complicate nonlinearities of flow field where acoustic wave propagates with a constant speed and has been extremely successful in understanding and predicting aerodynamically generated noise, but has been limited to subsonic cases where acoustic sources are contained in a small region. Curle[4] extended Lighthill's theory to account for stationary surface effects to the homogeneous wave equation. Ffowcs Williams and Hawkings[5] introduced a generalized function theory to moving boundaries. Their generalized function method solved a fundamental problem of moving boundaries.

In a transonic range where the acoustic wave propagates with variable speeds a different method has been developed using a nonlinear differential equation derived from the conservation of mass and momentum. This nonlinear differential equation will keep the nonlinear effects of local flow field, such as nonlinear wave propagation and local shock waves, and the nonlinear delocalization phenomenon can be explained with this nonlinear differential equation. The concept of a nonlinear sonic curve in a rotating frame is introduced first to explain the nonlinear shock wave propagation to a far-field and is compared to the existing experimental results with high-speed rotors.

In this paper, the Lighthill acoustic analogy is extended to a moving body in transonic speed using a generalized function theory and a nonlinear differential governing equation is developed to explain the delocalization phenomenon and also the delocalization Mach number of rotating blades. The introduction of the concept of the delocalization Mach number is a useful concept in understanding and controlling the rotor high-speed impulsive noise and can easily explain the nonlinear wave propagation of rotor noise.

GOVERNING EQUATIONS

Governing equations for a moving body

The governing equations of noise generation and propagation in an open domain are derived from the conservations of mass and momentum:

Mass :

$$\frac{\partial \rho}{\partial t} + div\,\rho\vec{V} = 0 \tag{1}$$

Momentum :

$$\frac{\partial \vec{V}}{\partial t} + grad\left(\frac{V^2}{2}\right) - \vec{V} \times curl\vec{V} = -\frac{1}{\rho}\,gradP \tag{2}$$

where ρ is density, V is velocity, and P is pressure. By rearranging these equations and adding a constant term $a_0^2\nabla^2\rho$ in both sides, the following inhomogeneous wave equation can be obtained:

$$\frac{\partial^2 \rho}{\partial t^2} - a_0^2\nabla^2\rho = \nabla \cdot \left(\rho\vec{V}\,\vec{V} + p\tilde{I} - a_0^2\rho\tilde{I}\right) \tag{3}$$

where $\tilde{I}$ is an identity tensor. Equation (3) was first derived by Sir Lighthill [3], called "Lighthill equation." With this equation, acoustic waves are assumed to propagate to a far-field with a constant velocity a_0, the acoustic source is contained in a small region compared to a wavelength, and no solid boundary near the source region is assumed. This Lighthill equation means that the acoustic source (in right-hand side of the equation) is assumed to be known and the solution is easily available in a closed form, and therefore aerodynamically-generated acoustic problems, such as rotor noise [2], can be easily solved.

This Lighthill equation has been extended for the cases of moving boundaries, particularly in subsonic cases, with a generalized function theory by Ffowcs Williams and Hawkings [5]. As shown by many mathematicians, the solution of the wave equation with a stationary boundary was obtained with several different methods, including a Green's function theory.

A body surface is described by $f(x,y,z,t) = 0$. That is, $\partial f / \partial t + \vec{V}_s \cdot \vec{n} |\nabla f| = 0$, where $\vec{V}_s \cdot \vec{n}$ is a normal component of body surface velocity. When the boundary is stationary, $\vec{V}_s \cdot \vec{n} = 0$. The solution of a wave equation is known as Kirchhoff formulation, which is easily available from any mathematics book, as follows:

$$\phi(\vec{x}, t) = \iint_S \left[\frac{1}{4\pi r} \frac{\partial \phi}{\partial n} \right]_\tau ds + div \iint_S \left[\frac{\phi \vec{n}}{4\pi r} \right]_\tau ds \tag{4}$$

where subscriptions s and τ means body surface and retarded time, respectively. Retarded time is defined as $t - r / a_0 \rho_1$, where r is the distance between an observer and a source.

For a moving boundary, many mathematicians have tried to extend this Kirchhoff formulation with a Green's function theory with a delta function without much success, due to a fact that a moving body can not be easily represented by the integral of delta functions. The solution of the wave equation with moving boundaries was first obtained by Ffowcs Williams and Hawkings [5] using a generalized function theory.

The conservation of mass becomes

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i} \overline{\rho u_i} = \rho_0 v_i \delta(f) \frac{\partial f}{\partial x_i} \tag{5}$$

The conservation of momentum becomes

$$\frac{\partial \overline{\rho u_i}}{\partial t} + \frac{\partial}{\partial x_j} \left(\overline{\rho u_i u_j} + \overline{P_{ij}} \right) = P'_{ij}\, \delta(f) \frac{\partial f}{\partial x_j} \tag{6}$$

where $P_{ij}^{'} = P_{ij} - P_0 \delta_{ij}$, $\overline{\rho}$ means $\rho_2 H + \rho_1 (1 - H)$, ρ_2 means the value of density outside of the boundary, ρ_1 means the value of density inside of the boundary, $H(f)$ is a Heaviside function, and $\delta(f)$ is a delta function. These conservation equations of mass and momentum can be rearranged as follows,

$$\left(\frac{\partial^2}{\partial t^2} - a_0^2 \frac{\partial^2}{\partial x_i^2} \right) \left(\overline{\rho - \rho_0} \right) = \frac{\partial^2 \overline{T_{ij}}}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i} \left(P'_{ij}\, \delta(f) \frac{\partial f}{\partial x_j} \right) + \frac{\partial}{\partial t} \left(\rho_0 v_i \delta(f) \frac{\partial f}{\partial x_i} \right) \tag{7}$$

where the generalized function $T_{ij} = \rho u_i u_j + P_{ij} - a_o^2 (\rho - \rho_0) \delta_{ij}$ called Lighthill stress tensor outside any surfaces, and is zero within them. The generalized function is formed with the aid of Heaviside's function $H(f)$ defined to be unity where $f > 0$ and zero where $f < 0$. Equation (7) is equivalent to the Lighthill equation, Eq. (3), for a moving boundary and the solution of this equation can be used for aeroacoustic problems, such as rotor noise or fan noise. The generalized function method shows the relation between aerodynamic sound and the scalar wave field of a moving surface in an arbitrary motion.

The general solution of this equation was first obtained by Ffowcs Williams and Hawkings [5] as follows:

$$4\pi a_0^2 \rho'(\vec{X}, t) = \frac{\partial^2}{\partial X_i \partial X_j} \iiint_V \left[\frac{T_{ij}}{r|1 - M_r|} \right]_\tau dV(\vec{\eta}) - \frac{\partial}{\partial X_i} \iint_S \left[\frac{P'_{ij} n_j}{r|1 - M_r|} \right]_\tau dS(\vec{\eta}) + \frac{\partial}{\partial t} \iint_S \left[\frac{\rho_o v_n}{r|1 - M_r|} \right]_\tau dS(\vec{\eta}) \tag{8}$$

where a bracket inside of the integral in the left-hand side means that the value is measured at the retarded time. To allow for the strength of the sources corresponding to a moving surface to be specified in a coordinate system moving with the surface, Lagrangian coordinates η are introduced which move with the sources. With this, a wave equation with moving boundaries is mathematically equivalent to an inhomogeneous wave equation in an infinite domain, where the right hand side is expressed in terms of different mathematical singularities. And a moving surface is acoustically equivalent to the distribution of mathematical monopoles and dipoles, and the flow field surrounding to a moving surface is acoustically equivalent to quadrupole sources.

Governing equations for a rotating blade in a transonic range

The Lighthill acoustic formulation, Eq. (3), and the subsequent extension to a moving surface (Eq. (7)) are based on linear cases at subsonic speeds. However, when a rotor blade speed reaches a transonic speed, these equations are extremely difficult to use, due to a fact that all nonlinear terms are moved to the right-hand side of the governing equation and these right-hand side terms are assumed to be known in priori. So, in order to understand the nonlinear effects such as nonlinear wave propagation, a nonlinear differential equation should be derived from the conservations of mass and momentum with keeping all nonlinear terms in the left-hand side of the governing equation.

The conservation of mass can be expressed as follows in the framework of the potential flow:

$$\frac{\partial \rho}{\partial t} + \nabla\phi \cdot \nabla\rho + \rho\nabla^2\phi = 0 \tag{9}$$

The conservation of momentum can be expressed as

$$\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + \frac{a^2}{\gamma - 1} = 0 \tag{10}$$

With isentropic flow assumption, rearrangement of these two equations becomes as follows:

$$\frac{\partial^2 \phi}{\partial t^2} - a^2\nabla^2\phi + 2\nabla\phi\cdot\nabla(\frac{\partial \phi}{\partial t}) + \frac{1}{2}\nabla\phi\cdot\nabla\left\{(\nabla\phi)^2\right\} = 0 \tag{11}$$

The equation is expressed in terms of a space-fixed reference frame and can be changed to a rotating frame as follows:

$$\frac{\partial \phi}{\partial t'} = \frac{\partial \phi}{\partial t} + \overrightarrow{V}'\cdot\nabla\phi = \frac{\partial \phi}{\partial t} + \left(\frac{\partial \overrightarrow{r}}{\partial t} + \overrightarrow{\Omega}\times\overrightarrow{r}\right)\cdot\nabla\phi \tag{12}$$

Thus, $\frac{\partial \phi}{\partial t'} = -\Omega\frac{\partial \phi}{\partial \Psi}$. Repeating this process,

$$\frac{\partial^2 \phi}{\partial t'^2} = -\overrightarrow{\Omega}\times\overrightarrow{r}\cdot\nabla\left(-\Omega\frac{\partial \phi}{\partial \Psi}\right) = \Omega^2\frac{\partial^2 \phi}{\partial \Psi^2} \tag{13}$$

With the coordinate transforming, Eq. (11) in the space-fixed coordinate system can be expressed in a coordinated system fixed to a rotating blade as follows:

$$\left\{1 - \left(\frac{\Omega r}{a_0}\right)^2 + (\gamma+1)\frac{\Omega}{a_0}\phi_\psi\right\}\phi_{\psi\psi} + \frac{r^2}{a_0}\left\{2\Omega\phi_r\phi_{r\psi} + 2\Omega\phi_z\phi_{z\psi}\right\}$$
$$+\frac{r^2}{a_0}\left\{a_0^2 + (\gamma-1)\Omega\phi_\psi\right\}\left\{\phi_{rr} + \frac{\phi_r}{r} + \phi_{zz}\right\} = 0 \tag{14}$$

This is a nonlinear differential equation which is transformed to a rotating system. However, this nonlinear differential equation has two distinctive characteristics, depending on the sign of the coefficient in the first term,

$$\left\{1 - \left(\frac{\Omega r}{a_0}\right)^2 + (\gamma+1)\frac{\Omega}{a_0}\phi_\psi\right\} \tag{15}$$

If this coefficient becomes negative, the local flow becomes supersonic and Eq. (14) becomes a wave equation, which means that a local shock wave will propagate to a far-field along the characteristic curves.

If this coefficient becomes positive, the local flow becomes subsonic and Eq. (14) becomes an elliptic equation, which means that a local disturbance will be spread out uniformly to all directions without propagation along any preferred direction or characteristics. So, the local shock waves may not be able to maintain its original shape to a far-field. For a special case where this coefficient, Eq. (15), becomes zero, this is called "nonlinear sonic cylinder." First, if we neglect the nonlinear term in Eq. (15) (third term) for simplicity, then the linear sonic circle can be described as $\Omega r = a_0$, as shown in Fig. 1.

Figure 1 shows a linear sonic circle, which is a perfect circle, without the nonlinear term in Eq. (15). However, if the nonlinear term is included, the nonlinear sonic curve will be formed, which is warped toward the rotor blade where the local transonic flow field is strong. So, when the nonlinear sonic curve is warped toward to the rotor blade and connected with the local shock waves on the blade, then this shock wave now propagates to a far-field along characteristic lines. Then the far-field acoustic waveform has the same form of blade local shock waves as shown in Fig. 2.

But, if this nonlinear sonic curve does not touch the local transonic flow on the blade, the far-field acoustic waveform is a collection of simple acoustic waves and becomes a smooth waveform as shown in Fig.3. The blade tip Mach number is called "delocalization Mach number," when the warped nonlinear sonic curve touches the blade local transonic flow field.

The delocalization phenomenon is a new concept for explaining the unexplainable experimental results for rotor high-speed impulsive noise and this is the first time to show the particular nonlinear wave propagation phenomenon (as shown in Figs. 2 and 3) with the nonlinear flow field equations. This new concept can now be used to design a rotor blade for less noise and vibration reduction.

CONCLUSIONS

A generalized formulation based on the generalized function theory in an inhomogeneous wave equation is presented for problems of acoustic fields with moving surfaces. The formulation makes use of the Heaviside function to establish generalized forms of the field variables. A nonlinear transonic aerodynamic equation is also formulated in a rotating frame and is used for explaining the far-field waveform changes from smooth to shock wave forms. The new concept of the delocalization phenomenon and a delocalization Mach number of rotors have been introduced and analyzed to explain the sudden changes of far-field rotor high-speed acoustic waveforms. This is a new concept and is a different approach compared to the widely-used Lighthill acoustic analogy to understand and control the generating mechanisms of high-speed impulsive noise.

The nonlinear transonic equation in a rotating frame also introduces a new concept of sonic curves, inside of which an elliptic characteristics will prevail and outside of which an hyperbolic characteristics will dominate. Therefore, local shock waves will propagate to a far-field without interruption, when the nonlinear sonic curves are connected with the transonic flow field on the blade. This phenomenon is called "delocalization" and the Mach number associated with this phenomenon is called "delocalization Mach number." This delocalization phenomenon has been observed in experimental results, in which smooth acoustic waveforms have been suddenly changed to shock waveforms at and above a certain Mach number. This new concept is the first time to explain the rotor nonlinear wave propagation in transonic range.

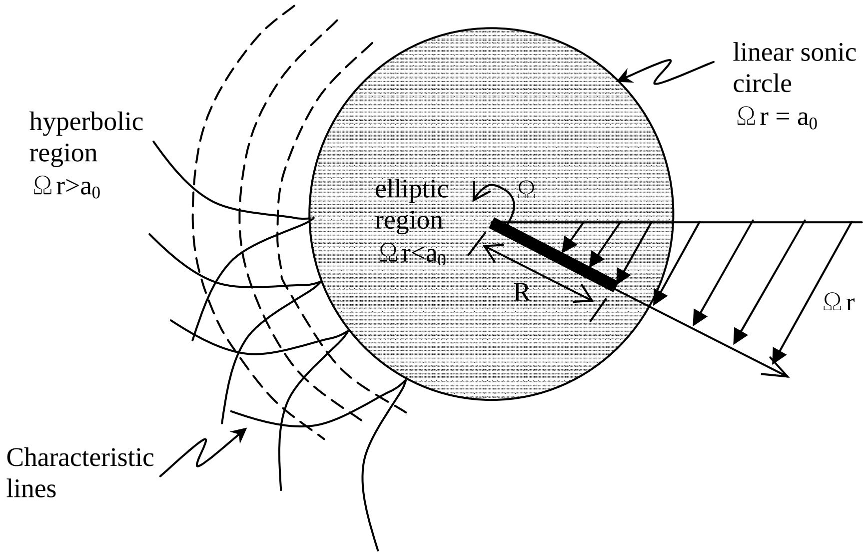

Figure 1. Characteristic lines and linear sonic circles for a rotating blade.

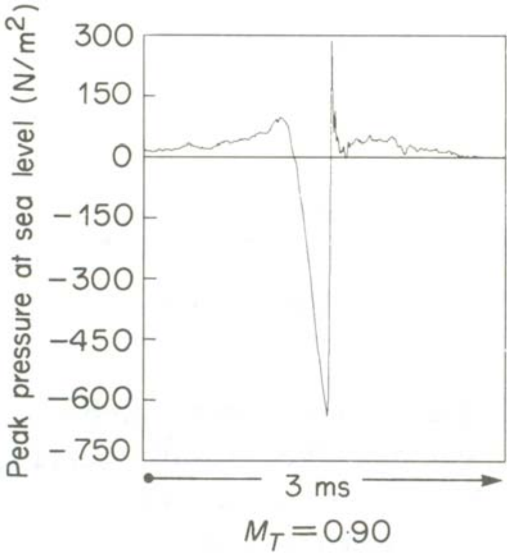

Figure 2. Shock waveform.

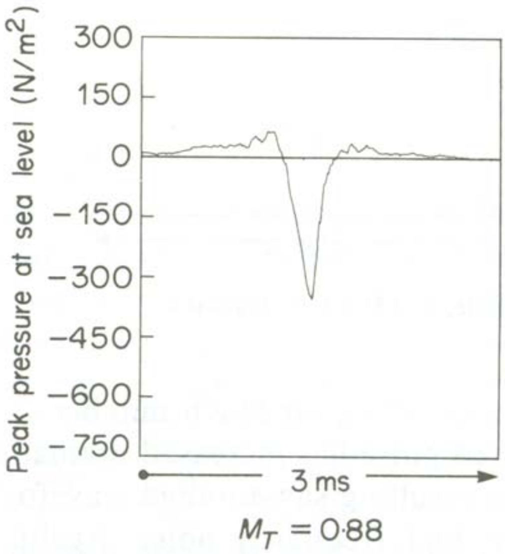

Figure 3. Smooth waveform.

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