NORMAL MODE ANALYSIS OF SECOND-ORDER PROJECTION METHODS FOR INCOMPRESSIBLE FLOWS

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A rigorous normal mode error analysis is carried out for two second-order projection type methods. It is shown that although the two schemes provide second-order accuracy for the velocity in $\mathbf{L}^2$-norm, their accuracies for the velocity in $\mathbf{H}^1$-norm and for the pressure in L^2-norm are different, and only the Gauge-Uzawa scheme introduced provides full second-order accuracy for all variable in their natural norms. The advantages and disadvantages of the normal mode analysis vs. the energy method are also elaborated.

GAUGE-UZAWA METHOD

Many projection type methods have been constructed to solve Navier-Stokes equations, and become the Representative solver in incompressible flows community. However they are still suffer from boundary layer, inconsistency, stability, or suboptimal accuracy, so on. Those difficulties are disappeared in Gauge-Uzawa method which has been studied in [3,5,7] and displays superior numerical performance. The goal of this paper is to prove fully 2nd order accuracy for velocity in $L^\infty(0,T;\mathbf{L}^2(\Omega))$ and $L^\infty(0,T;\mathbf{H}^1(\Omega))$ and for pressure in $L^\infty(0,T;L^2(\Omega))$ in normal mode space. We now introduce Gauge-Uzawa method:

Set initial values using a first-order gauge method with $\rho^0 = 0$ and repeat for $2 \le n \le N = [\frac{T}{\tau} - 1]$.

Step 1 Find $\widetilde{\mathbf{u}}^{n+1}$ as the solution of

$$\begin{cases} \dfrac{3\widetilde{\mathbf{u}}^{n+1} - 4\widetilde{\mathbf{u}}^{n} + \widetilde{\mathbf{u}}^{n-1}}{2\tau} + \nabla(2p^n - p^{n-1}) - \nu\triangle\widetilde{\mathbf{u}}^{n+1} = \mathbf{g}^{n+1}, \\ \widetilde{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{0}. \end{cases}$$

Step 2 Find ρ^{n+1} as the solution of

$$\begin{cases} -\triangle\rho^{n+1} = -\triangle\rho^n + \nabla\cdot\widetilde{\mathbf{u}}^{n+1}, \\ \partial_{\boldsymbol{\nu}}\rho^{n+1}|_\Gamma = 0. \end{cases}$$

Step 3 Update $\mathbf{u}^{n+1}$ and p^{n+1} by

$$\begin{aligned}\mathbf{u}^{n+1} &= \widetilde{\mathbf{u}}^{n+1} + \nabla(\rho^{n+1} - \rho^n) \\ p^{n+1} &= p^n - \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2\tau} + \nu\triangle\rho^{n+1}.\end{aligned} \tag{1}$$

THE MAIN RESULTS

We consider computational domain $\Omega = [-1, 1]\times[0, 2\pi]$ and $\mathbf{u} = (u, v)$ have periodic boundary conditions on $y = 0$ and $y = 2\pi$, it means $\mathbf{u}(x, 0) = \mathbf{u}(x, 2\pi)$. In addition, $\mathbf{u}(-1, y) = \mathbf{u}(1, y) = \mathbf{0}$. We now assume $(\mathbf{u}, p)(x, t) = \exp(\sigma t)(\bar{\mathbf{u}}, \bar{p})(x)$ to find the normal mode solution of Navier-Stokes equations. Then the symmetric solutions are

$$\begin{cases} \bar{u}(x) = \cos\mu x - \cos\mu\dfrac{\cosh kx}{\cosh k}, \\ \bar{v}(x) = \dfrac{\mu}{ik}\sin\mu x + \dfrac{1}{i}\cos\mu\dfrac{\sinh kx}{\cosh k}, \\ \bar{p}(x) = \dfrac{\sigma}{k}\cos\mu\dfrac{\sinh kx}{\cosh k}, \end{cases}$$

where $-\mu^2 = k^2 + \frac{\sigma}{\nu}$.

Since $\bar{v}(x)$ vanishes on boundary, we obtain

$$\mu\tan\mu + k\tanh k = 0.$$

We can find unique μ on each interval $I_s = (\frac{2s-1}{2}\pi, \frac{2s+1}{2}\pi)$. So the general normal mode solution of Navier-Stokes equations is

$$(\mathbf{u}, p)(x, y, t) = \sum_k\sum_s \alpha_{k,s}\exp(\sigma_{k,I_s}\cdot t)(\bar{\mathbf{u}}_{k,I_s}, \bar{p}_{k,I_s})(x)\exp(ky)$$

where $\alpha_{k,s}$ and $\beta_{k,s}$ are constants in the given initial velocity;

$$\mathbf{u}(x, y, 0) = \sum_k\sum_s \alpha_{k,s}\bar{\mathbf{u}}_{k,I_s}(x)\exp(ky).$$

We now start to find the normal mode solution of the Gauge-Uzawa method with a assumption

$$(\mathbf{u}^n, p^n) = \rho^n(\widehat{\mathbf{u}}, \widehat{p}).$$

Then we can get the symmetric solutions:

$$\begin{cases} \widehat{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}, \\ \widehat{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} + \dfrac{1}{ik}\dfrac{(\rho-1)^2}{2\rho-1}\dfrac{k^2+\widetilde{\mu}^2}{\widetilde{\mu}}\sin\widetilde{\mu}x, \\ \widehat{p}(x) = \dfrac{-\rho^2}{2\rho-1}\dfrac{\widetilde{\mu}^2+k^2}{k}\nu\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}. \end{cases}$$

Since $\widehat{v}(x)$ has 0 on boundary $x = \pm 1$, we obtain useful results

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = -\frac{(\rho-1)^2}{2\rho-1}\frac{k^2+\widetilde{\mu}^2}{\widetilde{\mu}}\tan\widetilde{\mu}.$$

$$-\widetilde{\mu}^2 = k^2 + \frac{(3\rho-1)(\rho-1)}{2\tau\rho^2\nu}, \quad \rho \in \left(\frac{1}{3}, 1\right)$$

$\widetilde{\mu}$ is unique in each $I_s = (\frac{2s-1}{2}\pi, \frac{2s+1}{2}\pi)$. If we consider fixed interval I_s and fixed k, $\rho \to 1$ as $\tau \to 0$. So we can get the following result

$$\left\|\mathbf{u}(t^{N+1}) - \mathbf{u}^{N+1}\right\|_{\mathbf{L}^\infty} + \left\|p(t^{N+1}) - p^{N+1}\right\|_{\mathbf{L}^\infty} + \left\|\nabla\cdot(\mathbf{u}(t^{N+1}) - \mathbf{u}^{N+1})\right\|_{\mathbf{L}^\infty} \le \tau^2.$$

Finally, we extend this results via energy estimate to

$$\left\|\nabla(\mathbf{u}(t^{N+1}) - \mathbf{u}^{N+1})\right\|_{\mathbf{L}^\infty} \le \tau^2.$$

We note that it is possible $\rho \approx \frac{1}{2}$ which is unstable condition, if τ is not small enough. So we have to assume τ is small enough to hold the accuracy results.

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