SEQUENTIAL OPTIMALITY CONDITIONS FOR CONVEX SEMIDEFINITE VECTOR OPTIMIZATION PROBLEMS

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ABSTRACT

Many authors [1-7] have studied semidefinite scalar optimization problems since the probelems have many engineering applications and many kinds of optmization problems can be reduced to the problems. In this talk, we discuss optimality conditions for a convex semidefinite vector optimization problem which consists of more than two convex objective convex functions over a linear matrix inequality and a closed convex subset. Our optimality conditions, which can be applied without any constraint qualification, are expressed with sequences. So our optimality conditions can be called sequential optimality conditions.

FORMULATION

Now we consider the following convex semidefinite vector optimization problem(SDVP):

(SDVP) Minimize $f(x) := (f_1(x), \cdots, f_p(x))$

subject to $S = \{x \in C \mid \quad F_0 + \sum_{i=1}^m x_i F_i \succeq 0\}.$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, and for $i = 0, 1, \cdots, m$, $F_i \in S_n$, the space of $(n \times n)$ real symmetric matrices, and C is a closed convex subset of $\mathbb{R}^m$. The space S_n is partially ordered by the $L\ddot{o}wner$ order; that is, for $M, N \in S_n,\ M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = Tr[MN]$, where $Tr[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then

$$S^+ = \{\theta \in S_n \mid (\theta, Z) \geqq 0 \ \forall Z \in S\} = S.$$

Let $F(x) := F_0 + \sum_{i=1}^m x_i F_i$, $\hat{F}(x) = \sum_{i=1}^m x_i F_i$, $x = (x_1 \cdots, x_m) \in \mathbb{R}^m$. Then $\hat{F}$ is a linear operator from $\mathbb{R}^m \ to \, S_n$ and its dual is defined by

$$\hat{F}^*(Z) = (Tr[F_1 Z], \cdots, Tr[F_m Z])$$

for any $Z \in S_n$.

Now we give solution concepts for **(SDVP)**:

Definition (1) $\bar{x} \in S$ is said to be an efficient solution of **(SDVP)** if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -\mathbb{R}^p_+ \backslash \{0\},$$

where $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$.

We denote all the efficient solutions of **(SDVP)** by Eff**(SDVP)**.

(2) $\bar{x} \in S$ is called a properly efficient solution of **(SDVP)** if $\bar{x} \in S$ is an efficient solution of **(SDVP)** and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) < f_i(\bar{x})$.

We denote all the properly efficient solutions of **(SDVP)** by $PrEff$**(SDVP)**.

(3) $\bar{x} \in S$ is said to be a weakly efficient solution of **(SDVP)** if for any $x \in S$,

$$(f_1(x) - f_1(\bar{x}), \cdots, f_p(x) - f_p(\bar{x})) \notin -int\mathbb{R}^p_+,$$

where $int\mathbb{R}^p_+$ is the interior of $\mathbb{R}^p_+$.

We denote all the weakly efficient solutions of **(SDVP)** by $WEff$**(SDVP)**.

SEQUENTIAL OPTIMALITY CONDITIONS

Now we present sequential optimality theorems for **(SDVP)**without proofs.

Theorem 1. Let $\theta \in int\mathbb{R}^p_+$. Then the following are equivalent :

(i) $a \in Eff$**(SDVP)**.

(ii) there exists $u \in \partial(\theta f)(a)$ such that

$$-\begin{pmatrix} u \\ u^t a \end{pmatrix} \in cl \left(\bigcup_{Z \in T} epi(-h_Z)^* + \bigcup_{\mu \in \mathbb{R}^p_+} \Big[epi(\mu f)^* + (0, (\mu f)(a)) \Big] + epi\delta^*_c \right),$$

where $h_Z(x) = Tr[ZF(x)]$.

(iii) there exist $u \in \partial(\theta f)(a),\ \epsilon_n \geqq 0,\ Z_n \in T$, $\mu_n \in \mathbb{R}^p_+, w_n \in \partial_{\epsilon_n}(\mu_n f)(a), \rho_n \geqq 0, v_n \in N^{\rho_n}_c(a)$such that

$$u + \lim_{n\to\infty}(-\widehat{F}^*(Z_n) + w_n + v_n) = 0,$$
$$\lim_{n\to\infty} Tr[Z_n F(a)] = 0, \quad \text{and}$$
$$\lim_{n\to\infty} \epsilon_n = 0 = \lim_{n\to\infty} \rho_n.$$

(iv) there exist $u \in \partial(\theta f)(a),\ Z_n \in T,\ \mu_n \in \mathbb{R}^p_+,\ x_n \in \mathbb{C},\ s_n \in -\widehat{F}^*(Z_n) + \partial(\mu_n f)(x_n) + N_c(x_n)$ such that

$$u + \lim_{n\to\infty} s_n = 0,$$
$$\lim_{n\to\infty} \{-Tr[Z_n F(x_n)] + (\mu_n f)(x_n) - (\mu_n f)(a)\} = 0, \quad \text{and}$$
$$\lim_{n\to\infty} x_n = a.$$

Theorem 2. The following are equivalent :

(i) $a \in PrEff$**(SDVP)**.

(ii) there exist $\theta \in int\mathbb{R}^p_+,\ u \in \partial(\theta f)(a),\ Z_n \in T, \rho_n \geqq 0, v_n \in N_c^{\rho_n}(a)$ such that

$$\lim_{n\to\infty} (\widehat{F}^*(Z_n) - v_n) = u, \quad \text{and}$$
$$\lim_{n\to\infty} Tr[Z_n F(a)] = 0$$
$$\lim_{n\to\infty} \rho_n = 0.$$

Theorem 3. The followings are equivalent :

(i) $a \in WEff$**(SDVP)**.

(ii) there exist $\theta \in \mathbb{R}^p_+ \setminus \{0\},\ u \in \partial(\theta f)(a),\ Z_n \in T, \rho_n \geqq 0, v_n \in N_c^{\rho_n}(a)$ such that

$$\lim_{n\to\infty} (\widehat{F}^*(Z_n) - v_n) = u, \quad \text{and}$$
$$\lim_{n\to\infty} Tr[Z_n F(a)] = 0$$
$$\lim_{n\to\infty} \rho_n = 0.$$

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