ASYMPTOTIC ANALYSIS OF HIGH-CONTRAST PHONONIC CRYSTALS AND A CRITERION FOR THE BAND GAP OPENING

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ABSTRACT

We investigate the band-gap structure of the frequency spectrum for elastic waves in a high-contrast, two-component periodic elastic medium. We consider two-dimensional phononic crystals consisting of a background medium which is perforated by an array of holes periodic along each of the two orthogonal coordinate axes. In this paper we establish a full asymptotic formula for dispersion relations of phononic band structures. The main ingredients are integral equation formulations of the solutions to the harmonic oscillatory linear elastic equation and several theorems concerning the characteristic values of meromorphic operator-valued functions in the complex plane such as Generalized Rouché's theorem. We establish a connection between the band structures and the Dirichlet eigenvalue problem on the elementary hole. We also provide a criterion for exhibiting gaps in the band structure which shows that smaller the density of the matrix is, wider the band-gap is, provided that the criterion is fulfilled. This phenomenon was reported by Economou and Sigalas in [15] who observed that periodic elastic composites whose matrix has lower density and higher shear modulus compared to those of inclusions yield better open gaps. Our analysis in this paper agrees with this experimental finding.

INTRODUCTION

In the past decade there has been a steady growth of interest in the motion of elastic waves through inhomogeneous materials. The primary motive for these investigations has been the design of the so-called phononic band gap materials or phononic crystals. The most recent research in this field has focused on theoretical and experimental demonstration of band gaps in two-dimensional and three-dimensional structures constructed of high-contrast elastic materials arranged in a periodic array. This type of structure prevents elastic waves in certain frequency ranges from propagating and could be used to generate frequency filters with control of pass or stop bands, as beam splitters, as sound or vibration protection devices, or as elastic waveguides.

Let D be a connected domain with Lipschitz boundary lying inside the open square $]0, 2\pi[^2$. An important example of phononic crystals consists of a background elastic medium of constant Lamé parameters λ and μ which is perforated by an array of arbitrary-shaped inclusions $\Omega = \cup_{n\in\mathbb{Z}^2}(D + n)$ periodic along each of the two orthogonal coordinate axes in the plane. These inclusions have Lamé constants $\tilde{\lambda}$, $\tilde{\mu}$. The shear modulus μ of the background medium is assumed to be larger than that of the inclusion $\tilde{\mu}$. Then we investigate the spectrum of the self-adjoint operator defined by

$$\mathbf{u} \mapsto -\nabla \cdot (C\nabla \mathbf{u}) = -\sum_{j,k,l=1}^{2} \frac{\partial}{\partial x_j}\left(C_{ijkl} \frac{\partial u_k}{\partial x_l} \right), \tag{1}$$

which is densely define on $L^2(\mathbb{R}^2)^2$. Here the elasticity tensor C is given by

$$C_{ijkl} := \left(\lambda\, \chi(\mathbb{R}^2 \setminus \overline{\Omega}) + \widetilde{\lambda}\, \chi(\Omega) \right) \delta_{ij}\delta_{kl} + \left(\mu\, \chi(\mathbb{R}^2 \setminus \overline{\Omega}) + \widetilde{\mu}\, \chi(\Omega) \right) (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2}$$

where $\chi(\Omega)$ is the indicator function of Ω.

In this paper we adopt this specific two-dimensional model to understand the relationship between the contrast of the shear modulus (and the density) of the inclusions and the band gap structure of the phononic crystal.

By Floquet theory [2], the spectrum of the Lamé system with periodic coefficients is represented as a union of bands, called phononic band structure. Carrying out a band structure calculation for a given phononic crystal involves a family of eigenvalue problems, as the quasi-momentum is varied over the first Brillouin zone. The problem of finding the spectrum of (1) is reduced to a family of eigenvalue problems with quasi-periodicity condition, *i.e.*,

$$\nabla \cdot (C\nabla \mathbf{u}) + \omega^2 \mathbf{u} = 0 \text{ in } \mathbb{R}^2, \tag{3}$$

where $\mathbf{u}(x + n) = e^{i\alpha\cdot n}\mathbf{u}(x)$ for every $n \in \mathbb{Z}^2$. Here the quasi-momentum α varies over $[0, 2\pi[^2$. Each of these operators has compact resolvent so that its spectrum consists of discrete eigenvalues of finite multiplicity. We show that these eigenvalues are the characteristic values of meromorphic operator-valued functions that are of Fredholm type with index zero. This yields a new and natural approach to the computation of the band gap phononic structure which is based on a combination of boundary element methods and Muller's method [4] for finding complex roots of scalar equations. Efficiency of a similar approach for computing photonic band gaps has been demonstrated in [12,13]. We then proceed from the generalized Rouché's theorem to construct their complete asymptotic expressions as the Lamé parameter μ of the background goes to infinity. For $\alpha \neq 0$, we prove that the discrete spectrum of (3) accumulates near the Dirichlet eigenvalues of Lamé system in D as μ goes to infinity. We then obtain a full asymptotic formula for the eigenvalues with the leading order term of order μ^{-1} calculated explicitly. For the periodic case $\alpha = 0$, we establish a formula for asymptotic behavior of eigenvalues, but their limiting set is generically different from that for $\alpha \neq 0$. We also consider the case when $|\alpha|$ is of order $1/\sqrt{\mu}$ and derive an asymptotic expansion for the eigenvalues in this case. Not surprisingly, this formula tends continuously to the previous ones as $\alpha\sqrt{\mu}$ goes to zero or to infinity. We finally provide a criterion for exhibiting gaps in the band structure. As has been said, the existence of those spectral gaps implies that the elastic waves in those frequency ranges are prohibited from travelling through the elastic body. Our criterion shows that smaller the density of the matrix is, wider the band-gap is, provided that the criterion is fulfilled. This phenomenon was reported by Economou and Sigalas in [15] who observed that periodic elastic composites whose matrix has lower density and higher shear modulus compared to those of inclusions yield better open gaps.

Similar results for the photonic crystals were obtained by Hempel and Lienau in [20], where they dealt with conductivity equation with high contrast in two phase composites. See also

Friedlander [18]. Another related work is [8] which concerns the photonic band gap using the same method as the one in this work. A justification of the existence of elastic band gaps in periodic composite materials with strong heterogeneities has been recently provided by Ávila et al. in [10] by extending Bouchitté and Felbacq scalar homogenization approach [11] to the elasticity problem. We also mention works by Movchan and his collaborators [24,26,25,23]. To the best of our knowledge, our result on gap opening is achieved by using a method significantly different from those in the literature. All the other asymptotic results are new and have never been established elsewhere.

The main ingredients in deriving the results of this paper are the boundary integral equations and the theory of meromorphic operator-valued functions. Using integral representations of the solutions to the harmonic oscillatory linear elastic equation, we reduce this problem to the study of characteristic values of integral operators in the complex planes. Generalized Rouché's theorem and other techniques from the theory of meromorphic operator-valued functions are combined with careful asymptotic expansions of integral kernels to obtain full asymptotic expansions for eigenvalues. This method was first used in [9], and then successfully applied to obtain an asymptotic formula for the eigenvalues of Laplacian under singular perturbations [7] and high contrast asymptotics for the photonic crystals [8]. See also [6].

Results of this paper could be used to design an optimization tool based on layer potential techniques for the systematic design of band-gap elastic materials and structures. Since the limiting situation reduces to easy-to-calculate spectra, the idea would be to start with these spectra (as initial guess) and then compute the gradient of some target functional using our asymptotic expansions with respect to the contrast. Moreover, in order to optimize the position and width of these gaps, we only need to optimize the shape of the inclusion considering the (more simpler) limiting situation.

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