# ON THE LOCATION OF CRITICAL POINT FOR THE POISSON EQUATION

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**ABSTRACT**

The location of the unique critical point of $\Delta u = -1$ is investigated by conformal mapping method to show that if the domain is given by $r = 1 + \epsilon p(\theta)$, the critical point coincides with the center of mass up to the order of $\epsilon$ although the two do not exactly match in general as shown by simple examples.

## INTRODUCTION

Solutions of partial differential equations generally possess many interesting and useful characteristics. Among them, the critical points where the gradient vanishes would be one of the most important. In fact, enough information (e.g. number, location, nature, etc) on these points combined with a proper inspection of level sets provides a complete geometry and topology of the solution in two or three dimension [8].

However, one finds just a few research results on critical points of partial differential equations and most of them are concerned with the number or the nature of critical points for certain specific types of partial differential equations. (See e.g. [2], [12].) Among them, we briefly mention two articles [1], [11] for elliptic partial differential equations in two dimension. These are interesting and fine results but none of them except [4] discusses where the critical points are located. Thus, more research on this issue is desired for complete and accurate description of the solution.

In this paper, we concentrate on the location of the (unique) critical point of Poisson equation $\Delta u = -1$ in two-dimension which is one of the most fundamental equations in mathematical physics. By the method of conformal map in complex plane, we successfully calculate the location of the critical point and the asymptotic shape of level curves for the perturbed domain from a unit disk. We then verify that the critical point agrees with the center of mass of the domain up to the small perturbation parameter if the domain is of the form $r = 1 + \epsilon p(\theta)$. Moreover, we suggest an example to show that the critical point does not exactly coincide with the center of mass. Additionally, another illustration is provided to show the emergence of new critical points during domain perturbation. We end with a remark of application to vortex dynamics.

**KEYWORDS**

critical point, location, vortex, level curve, stagnation point

## ACKNOWLEDGEMENT

This work was supported by Korea Research Foundation Grant KRF-2002-070-C00004. The author is grateful to Prof. Jaeyoung Byeon at POSTECH for fruitful discussions.


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