C$^1$-STABLY EXPANSIVE SETS FOR FLOWS

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Abstract. Let $X$ be a $C^1$ vector field on a closed $C^\infty$ manifold $M$. We introduce the concept of $C^1$ stable expansivity for compact $X_t$-invariant set, and use a flow-version of Mane’s results (Lemma II.3 in "Mane, R. An ergodic closing lemma. Ann. of Math. (2) 116 (1982), no. 3, 503–540") on uniformly hyperbolic families of periodic linear differential equations in order to get the hyperbolicity for the $C^1$ stably expansive homoclinic class $H_X(\gamma)$ of a hyperbolic periodic orbit $\gamma$.

1. Main results

Let $X$ be a $C^1$ vector field on a closed $C^\infty$ manifold $M$, and $\mathcal{X}^1(M)$ be the set of $C^1$ vector field on $M$ endowed with $C^1$ topology. Denoted $\mathcal{E}(M)$ by the set of expansive vector fields on $M$. Given a vector field $X \in \mathcal{X}^1(M)$, we denote $\text{Sing}(X)$ by the set of singularities of $X$; $\text{PO}(X_t)$ by the set of periodic orbits of $X_t$; $\text{R}(X)$ by the chain recurrent set of $X$. Let $\gamma$ be a hyperbolic periodic orbit of $X_t$. $W^s(\gamma)$ and $W^u(\gamma)$ denote the stable and unstable manifolds of $\gamma$; $H_X(\gamma)$ denotes the transversal homoclinic class of $X$ associated with $\gamma$, i.e., $H_X(\gamma) = \overline{W^s(\gamma) \cap W^u(\gamma)}$; $C_X(\gamma)$ denotes the chain component of $X$ containing $\gamma$; $P_t$ denotes the linear Poincaré flow defined on the normal bundle to $X$ over $M - \text{Sing}(X)$. For these definitions as well as hyperbolicity and dominated splitting of the linear Poincaré flow, see [2, 19]. The following result is from [12].
For $X \in \mathcal{X}^1(M)$, $X \in \text{int} \mathcal{E}(M)$ if and only if $\text{Sing}(X) = \emptyset$, and $X$ satisfies Axiom A and quasi transversality condition.

Our aim is to generalize the above result to some subsystems.

Let $d$ be the distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$. Every $X \in \mathcal{X}^1(M)$ generates a $C^1$ flow $X_t : M \times \mathbb{R} \to M$, that is a $C^1$ map such that $X_t : M \to M$ is a diffeomorphism satisfying $X_0(x) = x$ and $X_{t+s}(x) = X_t(X_s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. Throughout of this paper we assume that $X \in \mathcal{X}^1(M)$ has no singularity.

Let $\Lambda \subset M$ be a compact $X_t$-invariant subset. The set $\Lambda$ is called expansive for $X_t$ if for any $\varepsilon > 0$ there is $\delta > 0$ with the property that if $d(X_s(x), X_{\alpha(s)}(y)) \leq \delta$ for all $s \in \mathbb{R}$, for a pair of points $x, y \in \Lambda$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X_s(x)$ where $|s| \leq \varepsilon$. The constant $\delta$ is called an expansive constant corresponding to $\varepsilon$ (with respect to $X_t$). When $\Lambda$ is expansive for $X_t$, we also say that the subsystem $X|_{\Lambda}$ is expansive.

It is known that expansivity is a consequence of hyperbolicity. Moreover expansive systems share many like-hyperbolic properties. Hyperbolicity is open in the sense that all nearby systems of hyperbolic systems are hyperbolic. This is not true for expansivity: they are sensitive to small perturbations. A simple counterexample is a rational rotation on the unit circle. With those in mind, we introduce the following.

**Definition 1.1.** Let $\Lambda \subset M$ be a compact $X_t$-invariant subset. The set $\Lambda$ is called $C^1$ stably expansive for $X_t$ if there exist a compact set $U$ containing $\Lambda$ and a $C^1$ neighbourhood $U(X)$ of $X$ such that

(i) $\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$, i.e., $\Lambda$ is locally maximal (with isolating block $U$);

(ii) For all $Y \in U(X)$, $Y|_{\Lambda_Y}$ is expansive, where $\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$ is the continuation of $\Lambda$.

Note that $M$ is $C^1$ stably expansive for $X_t$ if and only if $X \in \text{int} \mathcal{E}(M)$. In that case, we say $X_t$ is a $C^1$ stably expansive flow. Suspension of the Smale’s horseshoe is an example of $C^1$ stably expansive flow. We cite the following fact to illustrate how natural the condition (i) is (for more details, see [15, Theorem 7.4]).

**Fact:** Let $\Lambda$ be a locally maximal hyperbolic set for $X_t$ (with isolating block $U$). Then for every $\varepsilon > 0$ there exists a $C^1$ neighbourhood $U(X)$ of $X$ such that for all $Y \in U(X)$,

(i) $\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$ is hyperbolic for $Y$,,
(ii) \( X|_\Lambda \) is topologically conjugate to \( Y|_\Lambda \) (with a conjugate map \( h_Y : \Lambda \to \Lambda_Y \)),
(iii) \( d_0(h_Y, 1_{\Lambda_Y}) < \varepsilon \).

**Proposition 1.2.** Suppose that \( \Lambda \) is \( C^1 \) stably expansive for \( X_t \). Then there is a \( C^1 \) neighbourhood \( U(X) \) of \( X \) such that for \( Y \in U(X) \), every \( q \in \Lambda_Y \cap \text{PO}(Y_t) \) is hyperbolic.

As a consequence of Proposition 1.2, we obtain a characterization of \( C^1 \) stable expansivity on \( R(X) \).

**Corollary 1.3.** Let \( X \in X^1(M) \). Then the followings are mutually equivalent:
(i) \( R(X) \) is \( C^1 \) stably expansive;
(ii) \( R(X) \) is hyperbolic.

Now we are in position to state our main theorem.

**Theorem 1.4.** Let \( X \in X^1(M) \). Then \( H_X(\gamma) \) is \( C^1 \) stably expansive if and only if \( H_X(\gamma) \) is hyperbolic.

The case of diffeomorphisms for Theorem 1.4 was proved in [5]. To prove Theorem 1.4, our main tool is a flow-version of Mane’s results on uniformly hyperbolic families of periodic linear differential equations on Euclidean spaces (see [9]). As a consequence of Proposition 1.2 with that flow-version of Mane’s results, we obtain a dominated splitting of the linear Poincaré flow over \( H_X(\gamma) \). Note that results by Doering ([3]) and Liao ([8]) said that the hyperbolicity of \( H_X(\gamma) \) for \( P_t \) is equivalent to the hyperbolicity of \( H_X(\gamma) \) for the underlying flow \( X_t \).

2. **Uniformly hyperbolic families of periodic linear differential equations**

Denoted \( PEP(\mathbb{R}^N) \) by the set of all invertible periodic matrix function \( U = U(\cdot) : \mathbb{R} \to GL(\mathbb{R}^N) \) such that \( \|U(t)\| \leq Ne^{\beta t} \) for every \( t \in \mathbb{R} \) and some positive constants \( N, \beta \) (depend on \( U \)), i.e., the space of all exponentially bounded periodic evolution processes on \( \mathbb{R}^N \). Every \( U \) can be considered as fundamental matrix (or evolution matrix, or evolution process) of some linear differential equation \( x' = A(t)x, \quad x \in \mathbb{R}^N \), with periodic continuous matrix coefficient \( A(\cdot) \).

\( PEP(\mathbb{R}^N) \) becomes a Banach space with the following norm:

\[ \|U - V\| := \sup_{t \in \mathbb{R}, \alpha \in I} \\left\{ \mu > 0 : \exists C > 0, \|U^{(\alpha)}(t) - V^{(\alpha)}(t)\| \leq Ce^{\beta t}\right\}. \]
Matrix function $U(\cdot) \in PEP(\mathbb{R}^N)$ is called hyperbolic if $\mathbb{R}^N = E^s(U) \oplus E^u(U)$, where

$$E^s(U) = \{ v \in \mathbb{R}^N : \sup_{t \geq 0} \|U(t)v\| < +\infty \}$$

$$E^u(U) = \{ v \in \mathbb{R}^N : \sup_{t \geq 0} \|U(-t)v\| < +\infty \}.$$ 

In case of $U(\cdot) \in PEP(\mathbb{R}^N)$, Floque’s Theorem states that there is $L(\cdot) \in GL(\mathbb{R}^N)$ (same period with $U(\cdot)$) such that

$$U(t) = L(t)e^{tB},$$

where $B$ is a constant matrix.

A simple example shows that the limit of sequence of hyperbolic matrix functions is not necessarily hyperbolic. A stronger concept, uniform hyperbolicity, will guarantee hyperbolicity of the limit matrix function.

**Definition 2.1.** Family $\{U^\alpha, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ is called uniformly hyperbolic if there exist $N, \beta, \varepsilon > 0$ such that

(i) $U^\alpha$ is hyperbolic for every $\alpha \in I$;

(ii) $\|U^\alpha\| \leq Ne^{\beta t}$, for all $\alpha \in I, t \in \mathbb{R}$;

(iii) if a family $\{V^\alpha, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ satisfies $\|U - V\| \leq \varepsilon$ then $V^\alpha$ is hyperbolic for every $\alpha \in I$.

Note that in condition (iii) we do not assume that period of $U$ and $V$ are the same. Now we state a flow-version of [9, Lemma II.3].

**Theorem 2.2.** Let $\{U^\alpha, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ be a uniformly hyperbolic family of periodic evolution processes. Then there exist constants $K, T, \lambda > 0$ such that

(i) if $U^\alpha$ has minimum period $\pi(U^\alpha) \geq T$ then for any $T' \in \mathbb{R}$ and any partition $0 < t_1 < t_2 < \cdots < t_k = \pi(U^\alpha)T'$, $t_{i+1} - t_i \geq T$, we have

$$\prod_{i=0}^{k-1} \|U^\alpha(t_{i+1} - t_i)|_{E^s(U^\alpha)}\| \leq Ke^{-\lambda(t_{i+1} - t_i)},$$

$$\prod_{i=0}^{k-1} \|U^\alpha(-t_{i+1} + t_i)|_{E^s(U^\alpha)}\| \leq Ke^{-\lambda(t_{i+1} - t_i)};$$

(ii) For $\alpha \in I, t \in \mathbb{R}$,

$$\|U^\alpha(t)|_{E^s(U^\alpha)}\| \cdot \left\|\left(U^\alpha(t)\right)^{-1}|_{E^s(U^\alpha)}\right\| \leq e^{-2\lambda};$$
For $\alpha \in I$, $T' \in \mathbb{R}$ and any partition $0 < t_1 < t_2 < \cdots < t_k = \pi(U^{(\alpha)})T'$, $t_{i+1} - t_i \geq T$, we have
\[
\frac{1}{\pi(U^{(\alpha)})T'} \sum_{i=0}^{k-1} \log \|U^{(\alpha)}(t_{i+1} - t_i|_{E^s(U^{(\alpha)})})\| < -\lambda,
\]
\[
\frac{1}{\pi(U^{(\alpha)})T'} \sum_{i=0}^{k-1} \log \|U^{(\alpha)}(-t_{i+1} + t_i|_{E^u(U^{(\alpha)})})\| < -\lambda.
\]

**Remark 2.3.** Some parts of Theorem 2.2 can be proved using the similar arguments as in [9], but some parts need another techniques to prove them, e.g. *Uniform Boundedness Principle* from Functional Analysis.

**Remark 2.4.** If $X \in X^*(M)$ (i.e., every singular and periodic point is stably hyperbolic), then we can apply Theorem 2.2 with evolution matrices is the linear Poincaré flow $P_t$, and obtain a dominated splitting for $P_t$ on $PO(X_t)|_{\text{Sing}(X)}$. This was proved by Liao (see [8], [18]) with *completely different techniques*.

Theorem 2.2 has its own interest. Next corollary is used in this paper.

**Corollary 2.5.** If $H_X(\gamma)$ is $C^1$ stably expansive then the linear Poincaré flow $P_t$ admits a dominated splitting on $H_X(\gamma)$.

Similarly we have:

**Corollary 2.6.** If $C_X(\gamma)$ is $C^1$ stably expansive then the linear Poincaré flow $P_t$ admits a dominated splitting on $C_X(\gamma)$.

It seems that it is more convenient to see Theorem 2.2 from point of view of Ordinary Linear Differential Equations (bounded solutions, exponentially bounded evolution process, ordinary dichotomy, exponential dichotomy, . . . ).

**REFERENCES**


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