TETRAVALENT SYMMETRIC GRAPHS OF ORDER 9p

SONG-TAO GUO AND YAN-QUAN FENG

Abstract. A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. In this paper, we classify tetravalent symmetric graphs of order 9p for each prime p.

1. Introduction

Let G be a permutation group on a set Ω and α ∈ Ω. Denote by $G_\alpha$ the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is semiregular on Ω if $G_\alpha = 1$ for every α ∈ Ω and regular if G is transitive and semiregular. Throughout this paper, we consider undirected finite connected graphs without loops or multiple edges. For a graph X we use V(X), E(X) and Aut(X) to denote its vertex set, edge set, and automorphism group, respectively. For $u, v \in V(X)$, denote by $\{u, v\}$ the edge incident to u and v in X.

A graph X is said to be vertex-transitive if Aut(X) acts transitively on V(X). An s-arc in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of the graph X such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is called an arc for short and a 0-arc is a vertex. For a subgroup $G \leq \text{Aut}(X)$, a graph X is said to be ($G, s$)-arc-transitive and ($G, s$)-regular if G is transitive and regular on the set of s-arcs in X, respectively. A ($G, s$)-arc-transitive graph is said to be ($G, s$)-transitive if it is not ($G, s+1$)-arc-transitive. In particular, a ($G, 1$)-arc-transitive graph is simply called $G$-symmetric. A graph X is simply called s-arc-transitive, s-regular and s-transitive if it is ($\text{Aut}(X), s$)-arc-transitive, ($\text{Aut}(X), s$)-regular and ($\text{Aut}(X), s$)-transitive, respectively.

Arc-transitive or s-transitive graphs have received considerable attention in the literature. For example, s-transitive graphs of order np was classified in [3, 4, 23] depending on n=1, 2 or 3, where p is a prime. Li [13] showed that there exists an s-transitive graph of odd order if and only if $s \leq 3$. For the case of valency 4, Gardiner and Praeger [8, 9] characterized tetravalent...
symmetric graphs, and Li et al. [14] classified vertex-primitive tetravalent s-transitive graphs. The classification of tetravalent s-transitive Cayley graphs on abelian groups was given by Xu and Xu [25]. We may deduce a classification of tetravalent 1-regular Cayley graphs on dihedral groups from [12, 18, 21, 22]. Zhou [31] gave a classification of tetravalent 1-regular graphs of order $pq$ for $p, q$ primes. Recently, Zhou [29] classified tetravalent s-transitive graphs of order $4p$, and Zhou and Feng [30] classified tetravalent s-transitive graphs of order $2p^2$. In this paper we classify tetravalent s-transitive graphs of order $9p$.

Throughout the paper we denote by $C_n$ and $K_n$ the cycle and the complete graph of order $n$, respectively. Denote by $Z_n$ the cyclic group of order $n$, by $Z_n^*$ the multiplicative group of $Z_n$ consisting of numbers coprime to $n$, by $D_{2n}$ the dihedral group of order $2n$, and by $F_n$ the Frobenius group of order $n$.

2. Preliminary results

For a subgroup $H$ of a group $G$, denote by $C_G(H)$ the centralizer of $H$ in $G$ and by $N_G(H)$ the normalizer of $H$ in $G$.

**Proposition 2.1** ([11, Chapter I, Theorem 4.5]). The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(H)$ of $H$.

The following proposition is due to Burnside.

**Proposition 2.2** ([19, Theorem 8.5.3]). Let $p$ and $q$ be primes, and let $m$ and $n$ be non-negative integers. Then every group of order $p^m q^n$ is solvable.

Let $G$ be a permutation group on a set $\Omega$. The size of $\Omega$ is called the degree of $G$ acting on $\Omega$.

**Proposition 2.3** ([6, Corollary 3.5B]). Every transitive permutation group of prime degree $p$ is either 2-transitive or solvable with a regular normal Sylow $p$-subgroup.

The following proposition is about the permutation group of degree $p^2$ for $p$ a prime.

**Proposition 2.4** ([28, Proposition 1]). Any transitive group of degree $p^2$ has a regular subgroup.

For a finite group $G$ and a subset $S$ of $G$ such that $1 \not\in S$ and $S = S^{-1}$, the Cayley graph $\text{Cay}(G, S)$ on $G$ with respect to $S$ is defined to have vertex set $V(\text{Cay}(G, S)) = G$ and edge set $E(\text{Cay}(G, S)) = \{ \{ g, sg \} \mid g \in G, s \in S \}$. Clearly, a Cayley graph $\text{Cay}(G, S)$ is connected if and only if $S$ generates $G$. Furthermore, $\text{Aut}(G, S) = \{ \alpha \in \text{Aut}(G) \mid S^\alpha = S \}$ is a subgroup of the automorphism group $\text{Aut}(\text{Cay}(G, S))$. Given a $g \in G$, define the permutation $R(g)$ on $G$ by $x \mapsto xg, x \in G$. Then $R(G) = \{ R(g) \mid g \in G \}$, called the right regular representation of $G$, is a permutation group isomorphic to $G$. The
Cayley graph is vertex-transitive because it admits the right regular representation $R(G)$ of $G$ as a regular group of automorphisms of $Cay(G, S)$. A Cayley graph $Cay(G, S)$ is said to be normal if $R(G)$ is normal in $Aut(Cay(G, S))$. A graph $X$ is isomorphic to a Cayley graph on $G$ if and only if $Aut(X)$ has a subgroup isomorphic to $G$, acting regularly on vertices (see [20]). For two subsets $S$ and $T$ of $G$ not containing the identity 1, if there is an $\alpha \in Aut(G)$ such that $S^\alpha = T$, then $S$ and $T$ are said to be equivalent, denoted by $S \equiv T$. We may easily show that if $S \equiv T$, then $Cay(G, S) \cong Cay(G, T)$ and $Cay(G, S)$ is normal if and only if $Cay(G, T)$ is normal.

**Proposition 2.5** ([26, Proposition 1.5]). A Cayley graph $Cay(G, S)$ is normal if and only if $Aut(Cay(G, S))_1 = Aut(G, S)$, where $Aut(Cay(G, S))_1$ is the stabilizer of 1 in $Aut(Cay(G, S))$.

From [1, Corollary 1.3], we have the following proposition.

**Proposition 2.6.** Let $X = Cay(G, S)$ be a connected tetravalent Cayley graph on a finite abelian group $G$ of odd order. Then $X$ is normal except for $G = Z_5$ and $X = K_5$.

For two subgroups $M$ and $N$ of a group $G$, $M \rtimes N$ stands for the semidirect product of $M$ by $N$. The next proposition characterizes the vertex stabilizers of connected tetravalent $s$-transitive graphs (see [14, Lemma 2.5] and [13, Theorem 1.1]).

**Proposition 2.7.** Let $X$ be a connected tetravalent $(G, s)$-transitive graph of odd order. Then $s \leq 3$ and the stabilizer $G_v$ of a vertex $v \in V(X)$ in $G$ is as follows:

1. $G_v$ is a 2-group for $s = 1$;
2. $G_v \cong A_4$ or $S_4$ for $s = 2$;
3. $G_v \cong Z_3 \times A_4$, $Z_3 \times S_4$, or $S_3 \times S_4$ for $s = 3$.

To introduce tetravalent symmetric graphs of order $3p$ for $p$ a prime, we define some graphs. Let $p > 3$ be a prime and let $Z_{3p} = Z_3 \times Z_p = \langle a \rangle \times \langle b \rangle$ be the cyclic group of order $3p$. Define $CA_{3p} = Cay(Z_{3p}, \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\})$. By the definition of $G(3p, 2)$ given in [23, Example 3.4], it is easy to see that $CA_{3p} \cong G(3p, 2)$ and $Aut(CA_{3p}) = Z_{3p} \rtimes Z_2^2$. The next proposition is about the classification of connected tetravalent symmetric graphs of order $3p$ (see [23, Theorem]).

**Proposition 2.8.** Let $p > 7$ be a prime and $X$ a connected tetravalent symmetric graph of order $3p$. Then $X \cong CA_{3p}$.

### 3. Graph constructions and isomorphisms

In this section we introduce connected tetravalent symmetric graphs of order $9p$ for $p$ a prime. The first example is the lexicographic product of $C_9$ and $2K_1$. 

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**Note:** The content is a natural representation of the document, maintained and adapted for clarity and readability.
Example 3.1. The lexicographic product $C_9[2K_1]$ is defined as the graph with vertex set $V(C_9) \times V(2K_1)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(C_9[2K_1])$, $u$ is adjacent to $v$ in $C_9[2K_1]$ if and only if $(x_1, x_2) \in E(C_9)$. Then $C_9[2K_1]$ is a connected tetravalent 1-transitive Cayley graph on the group $Z_9 \times Z_2$ and $\text{Aut}(C_9[2K_1]) = Z_9^2 \rtimes D_{18}$.

From [25, Example 3.2], we have the following example.

Example 3.2. Let $G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong Z_3 \times Z_3 \times Z_2$. The Cayley graph $G_{18} = \text{Cay}(G, \{ca, ca^{-1}, cb, cb^{-1}\})$ is 1-transitive and $\text{Aut}(G_{18}) = G \rtimes D_8$.

Xu and Xu [25] gave a classification of tetravalent arc-transitive Cayley graphs on finite abelian groups. The following example is extracted from [25, Example 3.2 and Theorem 3.5].

Example 3.3. Let $p \geq 3$ be a prime and $G = \langle a \rangle \times \langle b \rangle \cong Z_p \times Z_3$. Then the Cayley graph $CA_{(3,3p)}^1 = \text{Cay}(G, \{b, b^{-1}, ab, a^{-1}b^{-1}\})$ is 1-regular and $\text{Aut}(CA_{(3,3p)}^1) = G \rtimes Z_2^2$. Furthermore, if $p \equiv 3 \pmod{4}$, then there is only one connected tetravalent symmetric Cayley graph on the group $G$, that is, $CA_{(3,3p)}^1$, and if $p \equiv 1 \pmod{4}$ there are exactly two connected tetravalent symmetric Cayley graphs on the group $G$, that is, $CA_{(3,3p)}^1$ and $CA_{(3,3p)}^2$, where $CA_{(3,3p)}^2 = \text{Cay}(G, \{b, b^{-1}, ab^w, a^{-1}b^{-w}\})$ and $\text{Aut}(CA_{(3,3p)}^2) = G \rtimes Z_4$ with $w$ an element of order 4 in $Z_4^*$.

By [27, Theorems 1 and 3], there is only one connected tetravalent symmetric Cayley graph on the cyclic group of order $9p$ for each prime $p \geq 5$.

Example 3.4. Let $p \geq 5$ be a prime and $G = \langle a \rangle \times \langle b \rangle \cong Z_9 \times Z_p$. The unique connected tetravalent symmetric Cayley graph on $G$ is $CA_{9p} = \text{Cay}(G, \{ab, a^{-1}b^{-1}, a^{-1}b, ab^{-1}\})$, which is 1-regular and its automorphism group $\text{Aut}(CA_{9p}) = G \rtimes Z_2^2$.

Let $X = \text{Cay}(H, T)$ be a connected tetravalent symmetric Cayley graph on a non-abelian group $H$ of order 27. Then $(T) = H$, $T^{-1} = T$ and $|T| = 4$. By [7, Corollary 3.2], $X$ is normal, and hence $\text{Aut}(X)_1 = \text{Aut}(H, T)$ by Proposition 2.5. Since $|H| = 27$, we may assume that $T = \{x, x^{-1}, y, y^{-1}\}$. Thus, $\text{Aut}(H, T)$ is a 2-group and faithful on $T$, forcing that $\text{Aut}(H, T) \leq D_8$.

Since $X$ is symmetric, 4 does not divide $|\text{Aut}(H, T)|$. By the elementary group theory, there are two non-abelian groups of order 27:

$G_1(27) = \langle a, b | a^9 = b^3 = 1, b^{-1}ab = a^4 \rangle$;

$G_2(27) = \langle a, b, c | a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$.

If $H = G_1(27)$, then 4 divides $|\text{Aut}(H)|$ because each automorphism $\alpha \in \text{Aut}(H)$ has the following form:

$$\alpha : \begin{cases} a \mapsto a^ib^j, & (i, j) = 1, 0 \leq j \leq 2; \\
b \mapsto a^{3k}b, & 0 \leq k \leq 2. \end{cases}$$
This is impossible because \( 4 \mid |\text{Aut}(H,T)| \). Thus, \( H = G_2(27) \) and \( o(x) = o(y) = 3 \), where \( o(x) \) denotes the order of \( x \) in \( G_2(27) \). Since \( \langle x, y \rangle = H \) and \( [x, y] \in Z(H) = \langle c \rangle \), \( a, b \) and \( c \) have the same relations as do \( x, y \) and \( [x, y] \), which implies that the map \( a \mapsto x, b \mapsto y, c \mapsto [x, y] \) induces an automorphism of \( G_2(27) \). It follows that \( X \cong \text{Cay}(G_2(27), S) \), where \( S = \{ a, a^{-1}, b, b^{-1} \} \).

Clearly, the maps \( a \mapsto b, b \mapsto a, c \mapsto c \) and \( a \mapsto b, b \mapsto a^{-1}, c \mapsto c \) induce automorphisms of \( G_2(27) \), say \( \alpha_1 \) and \( \alpha_2 \), respectively. Then \( \alpha_1, \alpha_2 \in \text{Aut}(G_2(27), S) \) and \( \langle \alpha_1, \alpha_2 \rangle \cong D_8 \), forcing that \( X \) is symmetric. On the other hand, since \( \text{Aut}(G_2(27), S) \leq D_8 \), one has that \( \text{Aut}(G_2(27), S) = D_8 \) and \( \text{Aut}(X) = G_2(27) \rtimes D_8 \). Thus, we have the following example.

**Example 3.5.** Let \( G = G_2(27) = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle \) and \( S = \{ a, a^{-1}, b, b^{-1} \} \). Define

\[ \mathcal{G}_{27} = \text{Cay}(G, S) \]

Then \( \text{Aut}(\mathcal{G}_{27}) = G \rtimes D_8 \) and \( \mathcal{G}_{27} \) is the only connected tetravalent symmetric Cayley graph on non-abelian group of order 27.

Let \( X \) be a symmetric graph, and \( A \) an arc-transitive subgroup of \( \text{Aut}(X) \). Let \( \{ u, v \} \) be an edge of \( X \). Assume that \( H = A_u \) is the stabilizer of \( u \in V(X) \) and that \( g \in A \) interchanges \( u \) and \( v \). It is easy to see that the core \( H_A \) of \( H \) in \( A \) (the largest normal subgroup of \( A \) contained in \( H \)) is trivial, and that \( H_A^g \) consists of all elements of \( A \) which maps \( u \) to one of its neighbors in \( X \). By [16, 20], the graph \( X \) is isomorphic to the coset graph \( \text{Cos}(A,H,HgH) \), which is defined as the graph with vertex set \( \{ Ha \mid a \in A \} \), the set of right cosets of \( H \) in \( A \), and edge set \( \{ \{ Ha, Hda \} \mid a \in A, d \in HgH \} \). The valency of \( \text{Cos}(A,H,HgH) \) is \( |HgH|/|H| = |H : H \cap H^g| \), and \( \text{Cos}(A,H,HgH) \) is connected if and only if \( HgH \) generates \( A \). By right multiplication, every element in \( A \) induces an automorphism of \( \text{Cos}(A,H,HgH) \). Since \( H_A = 1 \), the induced action of \( A \) on \( V(\text{Cos}(A,H,HgH)) \) is faithful, and hence we may view \( A \) as a group of automorphisms of \( \text{Cos}(A,H,HgH) \).

From [14], one can see that, up to isomorphism, there is only one primitive tetravalent symmetric graph of order \( n \) if \( n = 45 \) or 153.

**Example 3.6.** Let \( G = \text{Aut}(A_6) \cong S_6 \rtimes \mathbb{Z}_2 \) and let \( P \) be a Sylow 2-subgroup of \( G \). By [5], \( P \) is a maximal subgroup of \( G \) and hence \( N_G(P) = P \). Let \( H \) be an elementary abelian 2-subgroup of \( P \) of order 8. Then \( N_G(H) \cong S_4 \rtimes \mathbb{Z}_2 \). Let \( d \) be an involution in \( N_G(H) \setminus P \). Define

\[ \mathcal{G}_{45} = \text{Cos}(G, P, PdP) \]

Then \( \mathcal{G}_{45} \) is a connected tetravalent 1-transitive graph and \( \text{Aut}(\mathcal{G}_{45}) \cong \text{Aut}(A_6) \).

**Example 3.7.** Let \( G = \text{PSL}(2,17) \) and let \( P = \langle a, b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle \cong D_{16} \) be a Sylow 2-subgroup of \( G \). By [5], \( P \) is a maximal subgroup of \( G \) and hence \( N_G(P) = P \). Let \( H = \langle a^4, b \rangle \). Then \( N_G(H) \cong S_4 \). Let \( d \) be an
involution in $N_G(H) \setminus P$. Define 

$$\mathcal{G}_{153} = \text{Cos}(G, P, PdP).$$

Then $\mathcal{G}_{153}$ is a connected tetravalent 1-transitive graph and $\text{Aut}(\mathcal{G}_{153}) \cong \text{PSL}(2, 17)$.

Since the automorphism groups of the graphs defined in Examples 3.1-3.7 are pairwise non-isomorphic, we have the following lemma.

**Lemma 3.8.** $C_9[2K_1], G_{153}, CA_1^{(3,3p)}, CA_2^{(3,3p)}, CA_{9p}^9, G_{27}, G_{45}$ and $G_{153}$ are connected pairwise non-isomorphic tetravalent symmetric graphs.

4. Classification

This section is devoted to classifying tetravalent symmetric graphs of order $9p$ for $p$ a prime. First we have the following lemma.

**Lemma 4.1.** Let $p$ be a prime greater than 3 and $G$ a non-abelian group of order $9p$. Then any connected tetravalent normal Cayley graph on $G$ cannot be symmetric.

**Proof.** Let $X = \text{Cay}(G, S)$ be a connected tetravalent normal Cayley graph. Then $\langle S \rangle = G$, $S^{-1} = S$ and $|S| = 4$. Since $|G| = 9p$, we may assume $S = \{x, x^{-1}, y, y^{-1}\}$, and since $X$ is normal, $\text{Aut}(G, S) = \text{Aut}(X)_1$ by Proposition 2.5.

Suppose to the contrary that $X$ is symmetric. Then $\text{Aut}(G, S)$ is transitive on $S$, forcing that $o(x) = o(y)$. Note that $p > 3$. By Sylow Theorem, $G$ has a normal Sylow $p$-subgroup, which means that $o(x) \neq p$ because $\langle S \rangle = G$. Denote by $Z(G)$ the center of $G$. From the elementary group theory, up to isomorphism, there are three non-abelian groups of order $9p$ for a prime $p > 3$:

- $G_1 = \langle a, b \mid a^p = b^3 = 1, b^{-1}ab = a^r \rangle$, where $r \in \mathbb{Z}_9^*$ and $o(r) = 3$;
- $G_2 = \langle a, b \mid a^p = b^3 = 1, b^{-1}ab = a^s \rangle$, where $s \in \mathbb{Z}_9^*$ and $o(s) = 9$;
- $G_3 = \langle a, b, c \mid a^p = b^3 = c^3 = [b, c] = [a, b] = 1, c^{-1}ac = a^t \rangle$, where $t \in \mathbb{Z}_9^*$ and $o(t) = 3$.

**Case 1:** $G = G_1$.

In this case, $Z(G) = \langle b^3 \rangle$ and $Z(G)$ is the unique subgroup of order 3 in $G$. Since $\langle S \rangle = G$, one has $o(x) \neq 3$ and hence $o(x) = o(y) = 3p$ or 9. Similarly, if $o(x) = 3p$, then $G = \langle S \rangle \subseteq Z(G) \times \langle a \rangle$, a contradiction. Thus, $o(x) = 9$ and $x, y$ have the form $a^ib^{3j+1}$ or $a^ib^{3j-1}$. Each automorphism $\alpha$ in $\text{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \leq i \leq p - 1; \\ b \mapsto a^jb^{3k+1}, & 0 \leq j \leq p - 1, 0 \leq k \leq 2. \end{cases}$$

Clearly, $\text{Aut}(G)$ is transitive on the set $\{[g, g^{-1}] \mid g \in G, o(g) = 9\}$. We may assume that $x = b$ and $y = a^ib^{3k+1}$. Since $a \mapsto a^i$, $b \mapsto b$ induces an automorphism of $G$, $S \equiv \{b, b^{-1}, ab^{3k+1}, (ab^{3k+1})^{-1}\}$. Note that every automorphism
of $G$ cannot map $b$ to $a^ib^{3k-1}$. It follows that $\text{Aut}(G, S) \leq \mathbb{Z}_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction.

**Case 2:** $G = G_2$.

Since $o(x) \neq p$, each element in $S$ has order $3$ or $9$, and since $\langle a, b^i \rangle$ is a metacyclic normal subgroup of order $3p$ containing all elements of order $3$, one has $o(x) \neq 3$. Thus, $o(x) = o(y) = 9$ and $x, y$ have the form $a^ib^{3j+1}$ or $a^ib^{3j-1}$.

Each automorphism $\alpha$ in $\text{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i, & 1 \leq i \leq p - 1; \\ b \mapsto ab, & 0 \leq j \leq p - 1. \end{cases}$$

Note that $a \mapsto a^i$, $b \mapsto b$ and $a \mapsto a, b^j \mapsto a^ib^j$ induce automorphisms of $G$. Then $S \equiv \{(a^{3k+1}, b^{3k+1}), (ab^{3k+1})^{-1}\}$. Since every automorphism of $G$ cannot map $b^i$ to $a^ib^{-i}$, one has $\text{Aut}(G, S) \leq \mathbb{Z}_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction.

**Case 3:** $G = G_3$.

Since $o(x) \neq p$, each element in $S$ has order $3p$ or $3$. Since $\langle a, b \rangle$ contains all elements of order $3p$ in $G$, one has $o(x) = 3$ because $\langle S \rangle = G$. Note that $Z(G) = \langle b \rangle$. Thus, $b, b^3 \notin S$, and $x, y$ have the form $a^ib^j$ or $a^ib^je^{-1}$ with $1 \leq i \leq p$ and $1 \leq j \leq 3$. Each automorphism $\alpha$ in $\text{Aut}(G)$ can be written as follows:

$$\alpha : \begin{cases} a \mapsto a^i & 1 \leq i \leq p - 1; \\ b \mapsto b^j & 1 \leq j \leq 2; \\ c \mapsto a^kb^lc & 0 \leq k \leq p - 1, 0 \leq l \leq 2. \end{cases}$$

Thus, we may assume that $x = c$, and since the map $a \mapsto a^i$, $b \mapsto b^j$, $c \mapsto c$ induces an automorphism of $G$, $S \equiv \{c, c^{-1}, abc, (abc)^{-1}\}$. Since every automorphism of $G$ cannot map $a^ib^j$ to $(a^ib^j)^{-1}$, one has $\text{Aut}(G, S) \leq \mathbb{Z}_2$. Thus, $\text{Aut}(G, S)$ cannot be transitive on $S$, a contradiction.

To state the main theorem, we introduce the so-called quotient graph. Let $X$ be a graph and let $G \leq \text{Aut}(X)$ be an arc-transitive subgroup on $X$. Assume that $G$ is imprimitive on $V(X)$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ is a complete block system of $G$. The block graph or quotient graph $X_G$ of $X$ relative to $\mathcal{B}$ is defined as the graph with vertex set the complete block system $\mathcal{B}$, and with the two blocks adjacent if and only if there is an edge in $X$ between those two blocks. Clearly, if $X$ is $G$-symmetric, then $X_G$ is $G/K$-symmetric, where $K$ is the kernel of $K$ on $\mathcal{B}$. For a normal subgroup $N$ of $G$, the set of the orbits of $N$ forms a complete block system of $G$. In this case we denote by $X_N$ the quotient graph of $X$ relative to the set of the orbits of $N$. The following is the main result of this paper.

**Theorem 4.2.** Let $p$ be a prime. Then any connected tetravalent symmetric graph of order $9p$ is isomorphic to one of the graphs in Table 1. Furthermore, all graphs in Table 1 are pairwise non-isomorphic.
By Lemma 3.8, all graphs in Table 1 are connected pairwise non-isomorphic tetravalent symmetric graphs. Let $X$ be a connected tetravalent symmetric graph of order $9p$. To finish the proof, it suffices to show that $X$ is isomorphic to one of the graphs listed in Table 1.

If $p \leq 7$, then by [17, 24], there are ten connected tetravalent symmetric graphs of order $9p$: two graphs for $p = 2$, two graphs for $p = 3$, four graphs for $p = 5$ and two graphs for $p = 7$. Thus, $X$ is isomorphic to $C_9[2K_2]$, $G_{18}$, $G_{27}$, $CA_{9p}^{1}$, $G_{45}$, $CA_{(3,3p)}^{1}$, $CA_{(3,3p)}^{1}$, $CA_{(3,3p)}^{1}$, $CA_{9p}$ or $CA_{(3,3p)}^{1}$.

Thus, in what follows one may assume that $p > 7$ and $X$ is not a normal Cayley graph. Then by Examples 3.3, 3.4 and Lemma 4.1, $X$ is isomorphic to $CA_{9p}$, $CA_{(3,3p)}^{1}$ or $CA_{(3,3p)}^{1}$.

Table 1.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$s$-transitive</th>
<th>$\text{Aut}(X)$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_9[2K_1]$</td>
<td>1-transitive</td>
<td>$\mathbb{Z}<em>2^2 \times D</em>{18}$</td>
<td>Example 3.1, $p = 2$</td>
</tr>
<tr>
<td>$G_{18}$</td>
<td>1-transitive</td>
<td>$(\mathbb{Z}_2^3 \times \mathbb{Z}_2) \times D_8$</td>
<td>Example 3.2, $p = 2$</td>
</tr>
<tr>
<td>$G_{27}$</td>
<td>1-transitive</td>
<td>$(\mathbb{Z}_2^3 \times \mathbb{Z}_3) \times D_8$</td>
<td>Example 3.5, $p = 3$</td>
</tr>
<tr>
<td>$G_{45}$</td>
<td>1-transitive</td>
<td>$\text{Aut}(A_6)$</td>
<td>Example 3.6, $p = 5$</td>
</tr>
<tr>
<td>$G_{153}$</td>
<td>1-transitive</td>
<td>$\text{PSL}(2,17)$</td>
<td>Example 3.7, $p = 17$</td>
</tr>
<tr>
<td>$CA_{9p}$</td>
<td>1-regular</td>
<td>$\mathbb{Z}_{9p} \rtimes \mathbb{Z}_2^2$</td>
<td>Example 3.4, $p \geq 5$</td>
</tr>
<tr>
<td>$CA_{(3,3p)}^{1}$</td>
<td>1-regular</td>
<td>$(\mathbb{Z}<em>3 \times \mathbb{Z}</em>{3p}) \rtimes \mathbb{Z}_2^2$</td>
<td>Example 3.3, $p \geq 3$</td>
</tr>
<tr>
<td>$CA_{(3,3p)}^{1}$</td>
<td>1-regular</td>
<td>$(\mathbb{Z}<em>3 \times \mathbb{Z}</em>{3p}) \rtimes \mathbb{Z}_4$</td>
<td>Example 3.3, $p \equiv 1(\text{mod } 4)$</td>
</tr>
</tbody>
</table>

Proof. By Lemma 3.8, all graphs in Table 1 are connected pairwise non-isomorphic tetravalent symmetric graphs. Let $X$ be a connected tetravalent symmetric graph of order $9p$. To finish the proof, it suffices to show that $X$ is isomorphic to one of the graphs listed in Table 1.

If $p \leq 7$, then by [17, 24], there are ten connected tetravalent symmetric graphs of order $9p$: two graphs for $p = 2$, two graphs for $p = 3$, four graphs for $p = 5$ and two graphs for $p = 7$. Thus, $X$ is isomorphic to $C_9[2K_2]$, $G_{18}$, $G_{27}$, $CA_{9p}^{1}$, $G_{45}$, $CA_{45}$, $CA_{(3,15)}^{1}$, $CA_{(3,15)}^{2}$, $CA_{63}$ or $CA_{(3,21)}^{1}$. Let $p > 7$ and assume that $X$ is a normal Cayley graph. Then by Examples 3.3, 3.4 and Lemma 4.1, $X$ is isomorphic to $CA_{9p}$, $CA_{(3,3p)}^{1}$ or $CA_{(3,3p)}^{1}$.

Thus, in what follows one may assume that $p > 7$ and $X$ is not a normal Cayley graph, that is, $A$ has no normal regular subgroup on $V(X)$. Then, to finish the proof it suffices to show that $X \cong G_{153}$.

Set $A = \text{Aut}(X)$ and let $A_v$ be the stabilizer of $v \in V(X)$ in $A$. Since $X$ is symmetric, either $A_v$ is a 2-group or $A_v \cong A_4$, $S_4$, $\mathbb{Z}_3 \times A_4$, $\mathbb{Z}_3 \times S_4$ or $S_3 \times S_4$ by Proposition 2.7. It follows that $|A| = 2^4 \cdot 3^4 \cdot p$ or $2^4 \cdot 3^2 \cdot p$ for some integer $t$. Since $p > 7$, every Sylow 2-subgroup of $A$ is also a Sylow 2-subgroup of a stabilizer of some vertex in $A$, implying that $A$ has no non-trivial normal 2-subgroups.

Suppose that $A$ has an intransitive minimal normal subgroup, say $N$. Since $|V(X)| = 9p$ and $|A| = 2^4 \cdot 3^3 \cdot p$ or $2^4 \cdot 3^2 \cdot p$, $N$ is either a non-abelian simple group, or an elementary abelian 3- or $p$-group. Let $B = \{B_1, B_2, \ldots, B_n\}$ be the set of orbits of $N$ and $K$ the kernel of $A$ acting on $B$. Then $N \leq K$. Let $m = |B_1|$. Then $mn = 9p$ with $1 < m, n < 9p$. The quotient graph $X_N$ has vertex set $B$ and $A/K \leq \text{Aut}(X_N)$. Moreover, assume that $B_1$ is adjacent to $B_2$ in $X_N$ with $v \in B_1$ and $u \in B_2$ being adjacent in $X$. Clearly, $X_N$ has valency 2 or 4.

**Case 1:** $X_N$ has valency 2.

In this case, $X_N$ is a cycle and $A/K \cong D_{2n}$. Since $X$ is symmetric, the induced subgraph $\langle B_1 \cup B_2 \rangle$ of $B_1 \cup B_2$ in $X$ is a union of several cycles of the
same length greater than 4, implying that $K_v$ is a 2-group and $K$ acts faithfully on $B_1$. Since $A/K \cong D_{2m}$, one has $|A| = 2^s mn = 2^s 9p$ for some integer $s$. This implies that if $A$ has a Hall $\{3,p\}$-subgroup, then it is regular on $V(X)$. Note that $mn = 9p$ with $1 < m, n < 9p$. Thus, $(B_1 \cup B_2) \cong C_{2m}, 3C_6, 3C_{2p}$ or $pC_6$.

Let $(B_1 \cup B_2) \cong C_{2m}$. Since Aut$(C_{2m}) \cong D_{4m}$, one has $Z_m \lesssim K \lesssim D_{2m}$, and since $A/K \cong D_{2m}$, $A$ has a normal subgroup of order $9p$, which is regular on $V(X)$ because $A_v$ is a 2-group. Thus, $A$ has a normal regular subgroup, a contradiction.

Let $(B_1 \cup B_2) \cong 3C_6$. Then $N$ has blocks of length 3 on $B_1$ and since $K$ acts faithfully on $B_1$, $N$ must be an elementary abelian 3-group and hence $K$ is a $\{2,3\}$-group. By Proposition 2.2, $K$ is solvable, and since $A/K \cong D_{2p}$, $A$ is solvable. Thus, $A$ has a Hall $\{3,p\}$-subgroup, say $G$, which is regular on $V(X)$. Since $N \leq G$, $G$ cannot be isomorphic to $G_1, G_2$ or $G_3$ as listed in Lemma 4.1. It follows that $G$ is abelian, and by Proposition 2.6, $X$ is a normal Cayley graph on $G$, a contradiction.

Now let $(B_1 \cup B_2) \cong 3C_{2p}$ or $pC_6$. Then $|B_1| = 3p$ and since $N$ is transitive on $B_1$, $N$ must be a non-abelian simple group, say $T$. By [5, pp. 12–14], $T$ is one of the following groups in Table 2.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Out</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5$</td>
<td>$2^2 \cdot 3 \cdot 5$</td>
<td>2</td>
</tr>
<tr>
<td>$A_6$</td>
<td>$2^3 \cdot 3^2 \cdot 5$</td>
<td>2</td>
</tr>
<tr>
<td>PSL$(2,7)$</td>
<td>$2^3 \cdot 3 \cdot 7$</td>
<td>2</td>
</tr>
<tr>
<td>PSL$(2,8)$</td>
<td>$2^3 \cdot 3^2 \cdot 7$</td>
<td>3</td>
</tr>
<tr>
<td>PSL$(2,17)$</td>
<td>$2^4 \cdot 3^2 \cdot 17$</td>
<td>2</td>
</tr>
<tr>
<td>PSL$(3,3)$</td>
<td>$2^4 \cdot 3^1 \cdot 13$</td>
<td>2</td>
</tr>
<tr>
<td>PSU$(3,3)$</td>
<td>$2^5 \cdot 3^1 \cdot 7$</td>
<td>3</td>
</tr>
<tr>
<td>PSU$(4,2)$</td>
<td>$2^6 \cdot 3^1 \cdot 5$</td>
<td>2</td>
</tr>
</tbody>
</table>

If $(B_1 \cup B_2) \cong 3C_{2p}$, then $N$ has a transitive action of degree 3, which is impossible because $N$ is a non-abelian simple group. Thus, $(B_1 \cup B_2) \cong pC_6$. Since $|A| = 2^s mn = 2^s 9p$ and $N$ is intransitive, $9p \nmid |N|$. Then by Table 2, one has $N \cong \text{PSL}(2,7)$. This is impossible because $p > 7$.

**Case 2:** $X_N$ has valency 4.

In this case, $K_v$ fixes the neighborhood of $v$ in $X$ pointwise. Thus, $K = N$ is semiregular on $V(X)$ and $A/N \lesssim \text{Aut}(X_N)$. Since $|V(X)| = 9p$, one has $N = \mathbb{Z}_p, \mathbb{Z}_3^3 \text{ or } \mathbb{Z}_3$.

Let $N \cong \mathbb{Z}_p$. Then the quotient graph $X_N$ has order 9. By Proposition 2.4, $A/N$ contains a regular subgroup, say $B/N$, on $V(X_N)$, that is, $X_N$ is a Cayley graph on $B/N$. It follows that $|B/N| = 9$ and hence $B/N$ is abelian. By
Proposition 2.6, $B/N \trianglelefteq A/N$ and hence $B \trianglelefteq A$. Thus, $B$ is a normal regular subgroup of $A$ on $V(X)$, a contradiction.

Let $N \cong \mathbb{Z}_2^4$. Then $X_N$ is a tetravalent $A/N$-symmetric graph of order $p$. Since $p > 7$, $X_N$ is not a complete graph, and hence $A/N$ has a normal regular Sylow $p$-subgroup by Proposition 2.3. This implies that $A$ has a normal regular subgroup, a contradiction.

Let $N \cong \mathbb{Z}_3$. Then $X_N$ is a connected tetravalent symmetric graph of order $3p$. Since $p > 7$, by Proposition 2.8 one has $X_N \cong CA_{3p}$. It follows that $A/N$ has a normal regular subgroup on $V(X_N)$ because $Aut(CA_{3p}) \cong \mathbb{Z}_{3p} \rtimes \mathbb{Z}_2^2$, which implies that $A$ has a normal regular subgroup on $V(X)$, a contradiction.

Now we may assume that $A$ has no intransitive minimal normal subgroup. Thus, every non-trivial normal subgroup of $A$ is transitive on $V(X)$. Again let $N$ be a minimal normal subgroup of $A$. Then $N$ is transitive on $V(X)$ and since $|V(X)| = 9p$, $N$ is a non-abelian simple group as listed in Table 2. Recall that $p > 7$ and either $|N_v| = 2^4$ or $|N_v| = 3 \cdot 2^3$, $3 \cdot 2^3$, $3^2 \cdot 2^4$, $3^2 \cdot 2^4$, or $3^2 \cdot 2^4$. It follows that $N \cong PSL(2, 17)$. Set $C = C_A(N)$, the centralizer of $N$ in $A$. Then $C \cap N = 1$ and $C$ is a $\{2, 3\}$-group. If $C \neq 1$, then $C$ is an intransitive normal subgroup of $A$ because $|V(X)| = 9p$, which is contrary to our assumption. Thus, $C = 1$ and $A = A/C \leq Aut(N)$ by Proposition 2.1. Since $N \cong PSL(2, 17)$, one has that $A = PSL(2, 17)$ or $PGL(2, 17)$, and the stabilizer $A_v$ is a Sylow 2-subgroup of $A$, which is maximal in $A$ by [5]. It follows that $A$ is primitive on $V(X)$, and by [14, Theorem 1.5] and Example 3.7, $X \cong \mathcal{G}_{153}$ and $A \cong PSL(2, 17)$. □

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