CROSS COMMUTATORS ON BACKWARD SHIFT INVARIANT SUBSPACES OVER THE BIDISK II

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ABSTRACT. In the previous paper, we gave a characterization of backward shift invariant subspaces of the Hardy space over the bidisk on which \([S_\phi S_\psi] = 0\) for a positive integer \(n \geq 2\). In this case, it holds that \(S_\psi^n = cI\) for some \(c \in \mathbb{C}\). In this paper, it is proved that if \([S_\phi S_\psi] = 0\) and \(\phi \in H^\infty(\Gamma_z)\), then \(S_\phi = cI\) for some \(c \in \mathbb{C}\).

1. Introduction

Let \(\Gamma^2\) be the 2-dimensional unit torus. We write \((z, w) = (e^{is}, e^{it})\) for variables in \(\Gamma^2 = \Gamma_z \times \Gamma_w\). Let \(L^2 = L^2(\Gamma^2)\) be the usual Lebesgue space on \(\Gamma^2\) with the norm
\[
\|f\|_2 = \left(\int_0^{2\pi} \int_0^{2\pi} |f(e^{is}, e^{it})|^2 \frac{dsdt}{(2\pi)^2}\right)^{1/2}.
\]
With the usual inner product, \(L^2\) is a Hilbert space. Let \(H^2 = H^2(\Gamma^2)\) be the Hardy space over \(\Gamma^2\). We denote by \(H^2(\Gamma_z)\) and \(H^2(\Gamma_w)\) the Hardy spaces on the unit circle in variables \(z\) and \(w\), respectively. We think of \(H^2(\Gamma_z)\) and \(H^2(\Gamma_w)\) as closed subspaces of \(H^2\). For each \(f \in H^2\), we can write \(f\) as
\[
f = \sum_{i=0}^{\infty} \oplus f_i(w)z^i, \quad f_i(w) \in H^2(\Gamma_w).
\]
Let \(P\) be the orthogonal projection from \(L^2\) onto \(H^2\). For a closed subspace \(M\) of \(L^2\), we denote by \(P_M\) the orthogonal projection from \(L^2\) onto \(M\). For a function \(\psi \in L^\infty\), the Toeplitz operator \(T_\psi\) on \(H^2\) is defined by \(T_\psi f = P(\psi f)\) for \(f \in H^2\). It is well known that \(T_\psi^* = T_\overline{\psi}\), and \(T_\psi^*T_\varphi = T_{\psi \varphi}\) for every \(\varphi \in H^\infty(\Gamma_z)\) and \(\psi \in H^\infty(\Gamma_w)\). A function \(f \in H^2\) is called inner if \(|f| = 1\) on \(\Gamma^2\) almost everywhere. A nonzero closed subspace \(M\) of \(H^2\) is

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called invariant if \( z M \subset M \) and \( w M \subset M \). In one variable case, the well known Beurling theorem [2] says that an invariant subspace \( M \) of \( H^2(\Gamma_z) \) has a form \( M = q(z)H^2(\Gamma_z) \), where \( q(z) \) is an inner function. In two variable case, the structure of invariant subspaces of \( H^2 \) is extremely complicated, see [3, 10].

Let \( M \) be an invariant subspace of \( H^2 \) with \( M \neq \{0\} \) and \( M \neq H^2 \). Then \( T_q^*(H^2 \ominus M) \subset H^2 \ominus M \) and \( T_w^*(H^2 \ominus M) \subset H^2 \ominus M \). In this paper, we write
\[
N = H^2 \ominus M.
\]

Usually, \( N \) is called a backward shift invariant subspace of \( H^2 \). See [1, 9] for studies of backward shift invariant subspaces over the unit circle \( \Gamma \).

For a function \( \psi \in L^\infty \), we denote by \( R_\psi \) the operator on \( M \) defined by \( R_\psi f = P_M(\psi f) \) for \( f \in M \). It holds \( R_\psi^* R_\psi = R_\psi \) and \( R_\psi = T_\psi | M \). We denote by \( [R_z, R_\psi] \) the cross commutator of \( R_z \) and \( R_\psi \), that is, \( [R_z, R_\psi] = R_z R_\psi - R_\psi R_z \). In [8], Mandrekar proved that \( [R_z, R_\psi] = 0 \) if and only if \( M \) is Beurling type, that is, \( M = qH^2 \) for some inner function \( q \) on \( \Gamma^2 \). This is a nice characterization of Beurling type invariant subspaces of \( H^2 \). More generally, in [4] the authors proved that \( [R_z, R_\psi] = 0 \) if and only if \( [R_{q_1(z)}, R_{q_2(w)}] = 0 \) for nonconstant functions \( q_1(z) \) and \( q_2(w) \). See [1, 9] for characterizations of \( H^2 \).

We define the operator \( S_\psi \) on \( N \) by \( S_\psi f = P_N(\psi f) \) for \( f \in N \). Then we have \( S_\psi^* S_\psi = T_\psi^* | N \). In [6], it is proved that \( [S_z, S^*_w] = 0 \) if and only if \( N \) has one of the following forms:

\[
\cdot \quad N = H^2 \ominus q_1(z)H^2,
\]
\[
\cdot \quad N = H^2 \ominus q_2(w)H^2,
\]
\[
\cdot \quad N = (H^2 \ominus q_1(z)H^2) \cap (H^2 \ominus q_2(w)H^2)
\]

for nonconstant one variable inner functions \( q_1(z) \) and \( q_2(w) \). In [7], it is shown that the condition \( [S_z, S^*_z] = 0 \) does not imply \( [S_z, S^*_w] = 0 \). In [5], the authors proved that for \( n \geq 2 \), \( [S_z^n, S^*_w] = 0 \) if and only if one of the following conditions holds:

(i) \([S_z, S^*_w] = 0\),

(ii) \(S_z^n S^*_w = 0\),

(iii) there exists a Blaschke product \( b(z) \) with
\[
b(z) = \prod_{j=1}^n \frac{z - \alpha_j}{1 - \overline{\alpha_j}z}, \quad 0 < |\alpha_j| < 1,
\]

where \( \alpha_i \neq \alpha_j \) for every \( i,j \) with \( i \neq j \) and \( \alpha_1^n = \alpha_2^n = \cdots = \alpha_n^n \) such that \( N \subset H^2 \ominus b(z)H^2 \).

In [7, Theorem 2.2], it is proved that (ii) holds if and only if either \( N \subset H^2(\Gamma_z) \) or \( N \subset H^2 \ominus z^nH^2 \). If \( N \subset H^2(\Gamma_z) \), then we have \([S_z, S^*_z] = 0\). Moreover, in [5] it is proved that if \([S_z^n, S^*_z] = 0\) and \([S_z, S^*_w] \neq 0\), then \( M \cap H^\infty(\Gamma_z) = \theta(z)H^\infty(\Gamma_z) \) for an inner function \( \theta(z) \), and \( z^n \in \mathbb{C} + \theta(z)H^\infty(\Gamma_z) \). This case, we have \( S_z^n = cI \) for some \( c \in \mathbb{C} \).

The purpose of this paper is to generalize the above phenomenon. Let \( \varphi(z) \in H^\infty(\Gamma_z) \) be a nonconstant function. Suppose that \([S_{\varphi(z)}, S^*_w] = 0 \) and
Lemma 2.3. 

$[S_z, S_w^*] \neq 0$. In Section 2, we prove that $M \cap H^\infty(\Gamma_z) \neq \{0\}$ and $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$. Hence by the Beurling theorem, $M \cap H^2(\Gamma_z) = \theta(z)H^2(\Gamma_z)$ for a nonconstant inner function $\theta(z)$. Thus we get $\theta(z)H^2 \subset M$. Write

$$M_\theta = M \oplus \theta(z)H^2.$$ 

We prove that $M_\theta \neq \{0\}$ and $T_{\varphi(z)}^* M_\theta \subset M_\theta$. In another word, $\varphi(z)N \subset N \oplus \theta(z)H^2$ holds. In Section 3, we study on the one variable Hardy space $H^2(\Gamma_z)$. Let $N_1, N_2$ be backward shift invariant subspaces of $H^2(\Gamma_z)$ satisfying $\{0\} \neq N_2 \subset \subset N_1 \neq H^2(\Gamma_z)$. It is proved that $\varphi(z)N_2 \subset N_2 \oplus (H^2(\Gamma_z) \oplus N_1)$ if and only if $\varphi(z) \in \mathbb{C} + (H^2(\Gamma_z) \oplus N_1)$. As applications of these facts, in Section 4 we prove that $\varphi(z) \in \mathbb{C} + \theta(z)H^\infty(\Gamma_z)$ and $S_\varphi = cI$ for some $c \in \mathbb{C}$.

2. Equivalent conditions for $[S_{\varphi(z)}, S_w^*] = 0$

Let $N$ be a backward shift invariant subspace of $H^2$ with $N \neq \{0\}$ and $N \neq H^2$, and let $\varphi(z) \in H^\infty(\Gamma_z)$ be a nonconstant function. We write operators $T_{\varphi}$ and $T_w^*$ on $H^2 = M \oplus N$ in the matrix forms as

$$T_{\varphi} = \begin{pmatrix} P_{M} T_{\varphi} | N & 0 \\ 0 & S_{\varphi} \end{pmatrix}, \quad T_w^* = \begin{pmatrix} 0 & 0 \\ P_N T_w^* | M & S_w^* \end{pmatrix} \text{ on } H^2 = \begin{pmatrix} M \oplus N \end{pmatrix}.$$ 

Let

$$A = P_M T_{\varphi} | N \quad \text{ and } \quad B = P_N T_w^* | M.$$ 

Since $T_{\varphi} T_w^* = T_w^* T_{\varphi}$ on $H^2$, we have

$$S_{\varphi} S_w^* = BA + S_w^* S_{\varphi}.$$ 

Hence we get the following.

**Lemma 2.1.** $[S_{\varphi}, S_w^*] = 0$ if and only if $BA = 0$.

It is not difficult to see that

$$\ker B = \{ f \in M : T_w^* f \in M \} = \{ f \in M \oplus wM : T_w^* f = 0 \} \oplus wM = (M \cap H^2(\Gamma_z)) \oplus wM$$

and

$$\overline{\text{range } A} = M \oplus \ker A^* = M \oplus \{ f \in M : T_{\varphi}^* f \in M \}.$$ 

Then by Lemma 2.1, we have the following.

**Lemma 2.2.** $[S_{\varphi}, S_w^*] = 0$ if and only if

$$M \oplus \{ f \in M : T_{\varphi}^* f \in M \} \subset (M \cap H^2(\Gamma_z)) \oplus wM.$$ 

**Lemma 2.3.** If $[S_{\varphi}, S_w^*] = 0$ and $[S_z, S_w^*] \neq 0$, then $M \cap H^2(\Gamma_z)$ is a nontrivial invariant subspace of $H^2(\Gamma_z)$. 
Proof. Since $M \neq H^2$, trivially $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$ holds. Suppose that $M \cap H^2(\Gamma_z) = \{0\}$. By Lemma 2.2,
\[ M \oplus \{ f \in M : T^*_w f \in M \} \subset wM. \]
Hence
\[ M \oplus wM \subset \{ f \in M : T^*_w f \in M \}. \]
Since $T^*_w T^*_\varphi = T^*_\varphi T^*_w$ on $H^2$, if $f \in M$ and $T^*_w f \in M$, then $T^*_\varphi (w^n f) = w^n T^*_w f \in M$ for every $n \geq 0$, so that by the above we get
\[ w^n(M \oplus wM) \subset \{ f \in M : T^*_\varphi f \in M \}. \]
Therefore
\[ M = \sum_{n=0}^{\infty} \oplus w^n(M \oplus wM) \subset \{ f \in M : T^*_\varphi f \in M \}. \]
Thus we get $T^*_\varphi M \subset M$. This shows that $\varphi(z) N \subset N$.

Let
\[ A = \{ \psi(z) \in H^\infty(\Gamma_z) : \psi N \subset N \}. \]
Then both functions 1 and $\varphi(z)$ are contained in $A$. For $\psi \in A$ and $h \in N$, we have
\[ N \ni T^*_\varphi (\psi h) = (T^*_\varphi \psi) h + \psi(0) T^*_\varphi h. \]
Hence $(T^*_\varphi \psi) N \subset N$, so that $T^*_\varphi A \subset A$. It is easy to see that $A$ is a weak-* closed subalgebra of $H^\infty(\Gamma_z)$. Let
\[ L = \{ f(z) \in H^1(\Gamma_z) : \int_0^{2\pi} f(e^{i\theta}) \overline{\varphi(e^{i\theta})} \frac{d\theta}{2\pi} = 0 \text{ for every } \psi(z) \in A \}. \]
Then $L$ is a closed subspace of $H^1(\Gamma_z)$. Since $T^*_\varphi A \subset A$ and $1 \in A$, we have $zL \subset L$.

Suppose that $L \neq \{0\}$. By the Beurling theorem, $L = q(z) H^1(\Gamma_z)$ for an inner function $q(z)$. Since $1 \in A$, $q(0) = 0$. Hence $\overline{\varphi}(z) \in H^\infty(\Gamma_z)$. Since $\varphi(z)^n \in A$ for $n \geq 1$,
\[ \int_0^{2\pi} e^{-i\theta} q(e^{i\theta}) \overline{\varphi(e^{i\theta})}^n e^{i\theta} h(e^{i\theta}) \frac{d\theta}{2\pi} = \int_0^{2\pi} q(e^{i\theta}) h(e^{i\theta}) \overline{\varphi(e^{i\theta})}^n \frac{d\theta}{2\pi} = 0 \]
for every $h(z) \in H^1(\Gamma_z)$. Hence $\overline{\varphi}(z) \overline{\varphi}(z)^n \in H^\infty(\Gamma_z)$ for every $n \geq 1$. By the Schneider theorem [11], we have $\varphi(z) \in H^\infty(\Gamma_z)$. This shows that $\varphi(z)$ is constant. Since we assumed that $\varphi(z)$ is nonconstant, this is a contradiction. Therefore $L = \{0\}$. Hence $A = H^\infty(\Gamma_z)$. Especially, we have $z \in A$ and $z N \subset N$. Then $T^*_z N = S_z$. Since $T^*_w N = S_w \text{ and } T^*_w T^*_z = T^*_w T^*_z$ on $H^2$, we have $S_z S_w = S_w S_z$. This is a desired contradiction. \[ \square \]

In the rest of this section, we assume that $M \cap H^2(\Gamma_z) \neq \{0\}$. Since $M \neq H^2$, $M \cap H^2(\Gamma_z) \neq H^2(\Gamma_z)$. By the Beurling theorem,
\[ M \cap H^2(\Gamma_z) = \theta(z) H^2(\Gamma_z) \]
Theorem 2.6. Let $f \in M_\theta$. Then $T^*_w f \in M_\theta$ if and only if $f \in wM_\theta$.

Proof. Suppose that $T^*_w f \in M_\theta$. Then

$$f - f(z,0) \in wM_\theta \subset M_\theta.$$  

Since $f \in M_\theta$, $f(z,0) \in M_\theta$. Since $M_\theta \cap H^2(\Gamma_z) = \{0\}$, $f(z,0) = 0$. Hence $f \in wM_\theta$. The converse is trivial. \qed

Let $P_\theta$ be the orthogonal projection from $H^2$ onto $H^2 \ominus \theta(z)H^2$, and $Q_\phi$ be the operator on $H^2 \ominus \theta(z)H^2$ defined by $Q_\phi f = P_\theta(\phi f)$ for $f \in H^2 \ominus \theta(z)H^2$.

We can write both operators $Q_\phi$ and $T^*_w \mid_{H^2 \ominus \theta(z)H^2}$ as

$$Q_\phi = \left( \begin{array}{cc} * & P_{M_\theta}T^*_w \cdot N \\ 0 & S_\phi \end{array} \right) \quad \text{on} \quad H^2 \ominus \theta(z)H^2 = \left( \begin{array}{c} M_\theta \\ N \end{array} \right).$$

and

$$T^*_w \mid_{H^2 \ominus \theta(z)H^2} = \left( \begin{array}{cc} * & 0 \\ P_NT^*_w \cdot M_\theta & S_w \end{array} \right) \quad \text{on} \quad H^2 \ominus \theta(z)H^2 = \left( \begin{array}{c} M_\theta \\ N \end{array} \right).$$

Let

$$A_\theta = P_{M_\theta}T^*_w \cdot N$$

and

$$B_\theta = P_NT^*_w \cdot M_\theta.$$

Lemma 2.5.  $[S_\phi, S^*_w] = 0$ if and only if $B_\theta A_\theta = 0$.

Proof. Let $f \in H^2 \ominus \theta(z)H^2 = M_\theta \oplus N$. We have $T^*_w (\varphi(z)f) = \varphi(z)T^*_w f$. Write

$$\varphi(z)f = Q_\phi f \oplus f_1 \in (M_\theta \oplus N) \ominus \theta(z)H^2.$$  

Since $T^*_w f_1 \in \theta(z)H^2$ and $T^*_w Q_\phi f \perp \theta(z)H^2$, we get $T^*_w Q_\phi f = Q_\phi T^*_w f$. Thus $Q_\phi T^*_w = T^*_w Q_\phi$ on $M_\theta \oplus N$. Similarly as Lemma 2.1, we can prove the assertion. \qed

The following is a slight generalization of [7, Theorem 4.4].

Theorem 2.6. The following conditions are equivalent;

(i) $[S_\phi, S^*_w] = 0$,
(ii) $M_\theta \ominus \{ f \in M_\theta : T^*_w f \in M_\theta \} \subset wM_\theta$,
(iii) $T^*_w M_\theta \subset M_\theta$. 


(iv) \( \varphi(z)N \subset N \oplus \theta(z)H^2 \).

**Proof.** By Lemma 2.4,

\[ \ker B_\theta = \{ f \in M_\theta : T_v^*f \in M_\theta \} = wM_\theta. \]

Also we have

\[ \overline{\text{range } A_\theta} = M_\theta \ominus \ker A_\theta = M_\theta \ominus \{ f \in M_\theta : T_v^*f \in M_\theta \}. \]

Hence by Lemma 2.5, we get (i) \( \Leftrightarrow \) (ii).

If (ii) holds, then

\[ M_\theta \ominus wM_\theta \subset \{ f \in M_\theta : T_v^*f \in M_\theta \}. \]

Hence for each \( n \geq 0 \), we have

\[ T^*_v (z) w^n (M_\theta \ominus wM_\theta) = w^n T^*_v (z) (M_\theta \ominus wM_\theta) \subset w^n M_\theta \subset M_\theta. \]

Since

\[ M_\theta = \sum_{n=0}^{\infty} w^n (M_\theta \ominus wM_\theta), \]

we have \( T^*_v M_\theta \subset M_\theta \). Thus we get (iii).

(iii) \( \Rightarrow \) (ii) is trivial.

It is not difficult to see that (iii) \( \Leftrightarrow \) (iv). \( \square \)

Suppose that \( [S_\varphi, S_w^*] = 0 \) and \( [S_z, S_w^*] \neq 0 \). Then we proved that

\[ \theta(z)H^2 \not\subset M \quad \text{and} \quad \varphi(z)(H^2 \ominus M) \subset (H^2 \ominus M) \oplus \theta(z)H^2. \]

Note that \( \theta(z)H^2 \) and \( M \) are invariant subspaces of \( H^2 \). Now we fix an inner function \( \theta(z) \). Here we have a question for which \( \varphi(z) \in H^\infty(\Gamma_z) \) satisfies the above condition. In the next section, we study a similar question in the one variable Hardy space \( H^2(\Gamma_z) \). In Section 4, we revisit on this question.

### 3. A theorem on the unit circle

In this section, we prove the following theorem.

**Theorem 3.1.** Let \( N_1, N_2 \) be backward shift invariant subspaces of \( H^2(\Gamma_z) \) with \( 0 \neq N_2 \subsetneq N_1 \neq H^2(\Gamma_z) \), and \( \varphi(z) \in N_1 \). Then

\[ \varphi(N_2 \cap H^\infty(\Gamma_z)) \subset N_2 \ominus (H^2(\Gamma_z) \subset N_1) \]

if and only if \( \varphi(z) = cP_{N_1}1 \) for some \( c \in \mathbb{C} \). In this case, if we define the operator \( S_\varphi \) on \( N_1 \) by \( S_\varphi f = P_{N_1}(\varphi f) \) for \( f \in N_1 \), then \( S_\varphi = cI \).

To prove the theorem, we need two lemmas which are not difficult to show.

**Lemma 3.2.** Let \( N \) be a backward shift invariant subspace of \( H^2(\Gamma_z) \). Then \( N \cap H^\infty(\Gamma_z) \) is dense in \( N \).

**Lemma 3.3.** Let \( N \) be a backward shift invariant subspace of \( H^2(\Gamma_z) \) with \( N \neq \{0\} \) and \( N \neq H^2(\Gamma_z) \). If \( \varphi \in H^2(\Gamma_z) \) is a nonconstant function, then \( \varphi(N \cap H^\infty(\Gamma_z)) \not\subset N \).
Proof of Theorem 3.1. By the Beurling theorem,
\[ H^2(\Gamma_z) \ominus N_1 = \theta H^2(\Gamma_z) \]
for some nonconstant inner function \( \theta \).

First, suppose that
\[ \varphi(N_2 \cap H^\infty(\Gamma_z)) \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1). \]
Since \( N_2 \neq \{0\} \), by Lemma 3.2 there exists \( h_1 \in N_2 \cap H^\infty(\Gamma_z) \) with \( h_1(0) = 1 \). Write
\[ \varphi = f_1 \oplus \theta g_1 \in N_2 \oplus (H^2(\Gamma_z) \ominus N_1) = N_2 \oplus \theta H^2(\Gamma_z). \]

Also for each \( h \in N_2 \cap H^\infty(\Gamma_z) \), we can write
\[ \varphi h = f \oplus \theta g \in N_2 \oplus \theta H^2(\Gamma_z). \]

When \( h(0) = 0 \), we shall prove that
\[ g(0) = 0. \]

By (3.2) and (3.1),
\[ T^*_z(f + \theta g) = T^*_z(f) + \theta T^*_z g + g(0)T^*_z \theta \]
\[ = (T^*_z f + g(0)T^*_z \theta) + \theta T^*_z g. \]

Note that \( T^*_z h \in N_2 \cap H^\infty(\Gamma_z) \) and \( T^*_z f + g(0)T^*_z \theta \perp \theta H^2(\Gamma_z) \). By the assumption, \( \varphi T^*_z h \in N_2 \oplus \theta H^2(\Gamma_z) \).

Thus \( g(0) = 0 \). Thus we get (3.3).

By (3.1) and (3.2),
\[ \varphi h = (f - h(0)f_1) \oplus \theta(g - h(0)g_1) \in N_2 \oplus \theta H^2(\Gamma_z). \]
Since \( (h - h(0))h_1(0) = 0 \), by (3.3) we get
\[ g(0) = h(0)g_1(0). \]

By (3.2) again,
\[ \varphi T^*_z h + h(0)T^*_z \varphi = T^*_z( f + \theta g) = (T^*_z f + g(0)T^*_z \theta), \]
so that
\[ \varphi T^*_z h = (h(0)T^*_z \varphi + T^*_z f + g(0)T^*_z \theta) \oplus \theta T^*_z g. \]
Since \( T^*_z h \in N_2 \cap H^\infty(\Gamma_z) \) and \( \varphi \perp \theta H^2(\Gamma_z) \), by the assumption we have
\[ -h(0)T^*_z \varphi + T^*_z f + g(0)T^*_z \theta \in N_2. \]
Similarly we have
\[ \varphi T_z^{n+2}h = \left( - (T_z^n h)(0)T_z^n \varphi - h(0)T_z^{n+2} \varphi + T_z^{n+2} f + g(0)T_z^{n+2} \theta \right) + \left( T_z^n g)(0)T_z^n \theta \right) \oplus \theta T_z^{n+2} g. \]

Repeating the same argument, we get
\[ \varphi T_z^{n+2}h = \left[ - \left( \sum_{j=0}^{n-1} (T_z^{n-j-1}h)(0)T_z^{n-j+1} \varphi \right) + T_z^{n+2} f \right. \]
\[ \left. + \left( \sum_{j=0}^{n-1} (T_z^{n-j}g)(0)T_z^{n-j} \theta \right) \right] \oplus \theta T_z^{n+2} g. \]

Since \( h \in N_2 \cap H^\infty(\Gamma_z) \), \( T_z^{n+2} h \in N_2 \cap H^\infty(\Gamma_z) \). Hence by (3.2) and (3.4),
\[ (T_z^{n+2} g)(0) = (T_z^{n+2} h)(0)g_1(0) \]
for every \( n \geq 0 \). This shows that \( g = g_1(0)h \). By (3.2), we obtain
\[ (\varphi - g_1(0) \theta) h = f \in N_2 \]
for every \( h \in N_2 \cap H^\infty(\Gamma_z) \). By Lemma 3.3, \( \varphi - g_1(0) \theta \) is constant. Write \( \varphi - g_1(0) \theta = c \). Since \( \varphi \in N_1 \), we have \( \varphi = cP_{N_1} \).

Next, suppose that \( \varphi = cP_{N_1} \). Then
\[ \varphi = cP_{N_1} = c(1 - \bar{\theta}(0) \theta). \]

Hence for \( f \in N_2 \cap H^\infty(\Gamma_z) \), we have
\[ \varphi f = cf - c \bar{\theta}(0) \theta f \in N_2 \oplus \theta H^2(\Gamma_z). \]

Thus we get \( \varphi (N_2 \cap H^\infty(\Gamma_z)) \subset N_2 \oplus (H^2(\Gamma_z) \ominus N_1) \).

**Corollary 3.4.** Let \( N_1, N_2 \) be backward shift invariant subspaces of \( H^2(\Gamma_z) \) with \( \{0\} \neq N_2 \subsetneq N_1 \neq H^2(\Gamma_z) \), and \( \varphi(z) \in L^\infty(\Gamma_z) \). Define the operator \( S_\varphi \) on \( N_1 \) by \( S_\varphi h = P_{N_1} \varphi h \) for \( h \in N_1 \). Then \( S_\varphi N_2 \subset N_2 \) if and only if
\[ \varphi \in C + H^2(\Gamma_z) \ominus N_1 \subset H^2(\Gamma_z) \ominus N_1 \subset H^2(\Gamma_z) \oplus (H^2(\Gamma_z) \ominus N_1). \]

**Proof.** Write \( H^2(\Gamma_z) \ominus N_1 = \theta H^2(\Gamma_z) \) for some inner function \( \theta \). Let
\[ \varphi = \varphi_1 \ominus \varphi_2 \ominus \theta \varphi_3 \in H^2(\Gamma_z) \ominus N_1 \oplus \theta H^2(\Gamma_z). \]

It is easy to see that
\[ P_{N_1} (\varphi_1 (N_2 \cap H^\infty(\Gamma_z)) ) \subset N_2 \]
and
\[ P_{N_1} (\theta \varphi_3 (N_2 \cap H^\infty(\Gamma_z))) = \{0\}. \]
Hence $S_\varphi N_2 \subset N_2$ if and only if $P_{N_1}(\varphi_2(N_2 \cap H^\infty(\Gamma_z))) \subset N_2$. By Theorem 3.1, $S_\varphi N_2 \subset N_2$ if and only if
\[
\varphi = \varphi_1 + cP_{N_1}1 + \theta \varphi_3 = \varphi_1 + c(1 - \overline{\theta(0)}) + \theta \varphi_3 = \varphi_1 + c + \theta(\varphi_3 - c\overline{\theta(0)}).
\]
This completes the proof. \qed

The following corollaries follow from Corollary 3.4 directly.

**Corollary 3.5.** Let $N_1, N_2$ be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq N_2 \subset N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. Then $\varphi N_2 \subset N_2 \oplus (H^2(\Gamma_z) \circ N_1)$ if and only if $\varphi \in \mathbb{C} + (H^2(\Gamma_z) \circ N_1)$.

**Corollary 3.6.** Let $N_1, N_2$ be backward shift invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq N_2 \subset N_1 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. If $\varphi N_2 \subset N_2 \oplus (H^2(\Gamma_z) \circ N_1)$, then $N_1 = N_2$ if and only if $\varphi \notin \mathbb{C} + (H^2(\Gamma_z) \circ N_1)$.

**Corollary 3.7.** Let $M_1, M_2$ be invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq M_1 \subset M_2 \neq H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. Then $T_\varphi^*(M_2 \circ M_1) \subset M_2 \circ M_1$ if and only if $\varphi \in \mathbb{C} + M_1$.

**Corollary 3.8.** Let $M_1, M_2$ be invariant subspaces of $H^2(\Gamma_z)$ with $\{0\} \neq M_1 \subset M_2 \subset H^2(\Gamma_z)$, and $\varphi(z) \in H^\infty(\Gamma_z)$. If $T_\varphi^*(M_2 \circ M_1) \subset M_2 \circ M_1$, then $\varphi \notin \mathbb{C} + M_1$ if and only if $M_2 = H^2(\Gamma_z)$.

4. The main theorem

As applications of the results in Sections 2 and 3, we prove the following.

**Theorem 4.1.** Let $N$ be a backward shift invariant subspace of $H^2$ with $N \neq \{0\}$ and $N \neq H^2$. Let $\varphi(z) \in H^\infty(\Gamma_z)$ be a nonconstant function. If $[S_\varphi, S_w] = 0$ and $[S_z, S_\varphi^*] \neq 0$, then $\varphi(z) - c \in M \cap H^\infty(\Gamma_z)$ for some $c \in \mathbb{C}$ and $S_\varphi = c1$.

**Proof.** By Lemma 2.3, $M \cap H^2(\Gamma_z) = \theta(z)H^2(\Gamma_z)$ for a nonconstant inner function $\theta(z)$. Since $\theta(z)H^2 \subset M$, as in Section 2 we write
\[
M_\theta = M \circ \theta(z)H^2.
\]
Since $[S_z, S_\varphi^*] \neq 0$, we have $M_\theta \neq \{0\}$. By Theorem 2.6,
\[
\varphi(z)N \subset N \circ \theta(z)H^2
\]
and
\[
T_\varphi^*M_\theta \subset M_\theta.
\]
To prove the assertion, we assume that
\[
\varphi(z) - c \notin \theta(z)H^\infty(\Gamma_z)
\]
for every $c \in \mathbb{C}$. We shall prove that $[S_z, S_\varphi^*] = 0$. This will be a desired contradiction. We consider two cases $\theta(0) = 0$ and $\theta(0) \neq 0$ separately.
Case 1. Suppose that $\theta(0) = 0$. If $\theta(z) = cz$ for some constant $c$ with $|c| = 1$, then it is easy to see that

$$M = \theta(z)H^2 + q(w)H^2$$

for either a nonconstant inner function $q(w)$ or $q(w) \equiv 0$. In this case, by [6] we have $[S_z, S_w^n] = 0$. So, we may assume that $\theta(z) = z\theta_1(z)$ for a nonconstant inner function $\theta_1(z)$. Then

$$(4.5) \quad H^2 \oplus \theta(z)H^2 = H^2(\Gamma_w) \oplus z(H^2 \oplus \theta_1(z)H^2).$$

We divide the proof into two subcases.

Subcase 1.1. Assume that $\theta_1(z)M_\theta \subset \theta(z)H^2$. Then $M_\theta \subset zH^2$. Hence $H^2(\Gamma_w) \subset N$. For each nonnegative integer $n$, let

$$L_n = \{f(z) \in H^2(\Gamma_z) : w^n f(z) \in N\}.$$  

Then $1 \in L_n$, $L_n$ is a nonzero closed subspace of $H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)$, and $T_z^* L_n \subset L_n$. By (4.2),

$$w^n \varphi(z) L_n \subset \varphi(z) N \subset N \oplus \theta(z)H^2,$$

so we have

$$\varphi(z) L_n \subset L_n \oplus \theta(z)H^2(\Gamma_z).$$

By (4.4) and Corollary 3.6, $L_n = H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)$. Hence

$$w^n (H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)) \subset N$$

for every $n \geq 0$. Therefore

$$H^2 \oplus \theta(z)H^2 = \sum_{n=0}^{\infty} w^n (H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)) \subset N.$$

By (4.1), $H^2 \oplus \theta(z)H^2 = M_\theta \oplus N$, so that $M_\theta = \{0\}$. This contradicts $[S_z, S_w^n] \neq 0$.

Subcase 1.2. Assume that $\theta_1(z)M_\theta \not\subset \theta(z)H^2$. By (4.5), for every $g \in M_\theta$ we can write

$$(4.6) \quad g = f_g(w) \oplus z h_g(z, w),$$

where $f_g \in H^2(\Gamma_w)$ and $h_g \in H^2 \oplus \theta_1(z)H^2$. Since $\theta_1(z)M_\theta \subset M$, we have

$$\theta_1(z) g = \theta_1(z) f_g(w) \oplus z \theta_1(z) h_g(z, w) \in M = M_\theta \oplus \theta(z)H^2,$$

so that $\theta_1(z) f_g(w) \in M_\theta$. Since $\theta_1(z)M_\theta \not\subset \theta(z)H^2$, $f_g(w) \not\equiv 0$ for some $g \in M_\theta$. Then $\{f_g(w) : g \in M_\theta\} \neq \{0\}$. Since $wM_\theta \subset M_\theta$, by (4.6) $\{f_g(w) : g \in M_\theta\}$ is a nonzero $T_w$-invariant subspace of $H^2(\Gamma_w)$. Hence there is a one variable inner function $q(w)$ such that

$$(4.7) \quad q(w)H^2(\Gamma_w) = \{f_g(w) : g \in M_\theta\}.$$
Since $\theta_1(z)\{f_g(w) : g \in M_\theta\} \subset M_\theta$, we have

$$\theta_1(z)q(w)H^2(\Gamma_w) \subset M_\theta. \tag{4.8}$$

If $q(w)$ is constant, then $\theta_1(z) \in M_\theta$ and

$$\theta(z)H^2(\Gamma_z) \subset \mathbb{C} \cdot \theta_1(z) + \theta(z)H^2(\Gamma_z) \subset M \cap H^2(\Gamma_z),$$

so that $\theta(z)H^2(\Gamma_z) \neq M \cap H^2(\Gamma_z)$. This is a contradiction. Hence $q(w)$ is nonconstant. By (4.6) and (4.7), we get

$$q(w)H^2(\Gamma_w) \perp M_\theta. \tag{4.9}$$

For each nonnegative integer $n$, let

$$L_n = \{ f(z) \in H^2(\Gamma_z) : \theta(z)H^2(\Gamma_z) : f(z)w^nq(w) \in M_\theta \}. \tag{4.10}$$

By (4.8), $\theta_1(z) \in L_n$. Since $zM_\theta \subset M_\theta \oplus \theta(z)H^2$, $L_n \oplus \theta(z)H^2(\Gamma_z)$ is an invariant subspace of $H^2(\Gamma_z)$. By (4.3), we have $T^*_zL_n \subset L_n$. By (4.4) and Corollary 3.8, $L_n = H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)$. Hence

$$w^nq(w)(H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)) \subset M_\theta$$

for every $n \geq 0$. Thus we get

$$q(w)(H^2 \oplus \theta(z)H^2) \subset M_\theta. \tag{4.11}$$

By (4.9), $H^2(\Gamma_w) \cap q(w)H^2(\Gamma_w) \subset N$. For each $\psi(w) \in H^2(\Gamma_w) \cap q(w)H^2(\Gamma_w)$, let

$$L_\psi = \{ f(z) \in H^2(\Gamma_z) : \theta(z)H^2(\Gamma_z) : f(z)\psi(w) \in N \}.$$

Then $1 \in L_\psi$, and in the same way as Subcase 1.1, $L_\psi$ is a nonzero closed subspace of $H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)$ such that $T^*_zL_\psi \subset L_\psi$ and $\varphi(z)L_\psi \subset L_\psi \oplus \theta(z)H^2(\Gamma_z)$. Hence by (4.4) and Corollary 3.6, $L_\psi = H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)$. Therefore

$$\psi(w)(H^2(\Gamma_z) \oplus \theta(z)H^2(\Gamma_z)) \subset N$$

for every $\psi(w) \in H^2(\Gamma_w) \cap q(w)H^2(\Gamma_w)$, and hence

$$H^2 \oplus \theta(z)H^2 \subset q(w)(H^2 \oplus \theta(z)H^2) \subset N. \tag{4.12}$$

Since $H^2 \oplus \theta(z)H^2 = M_\theta \oplus N$, by (4.10) and (4.11) we get

$$N = (H^2 \oplus \theta(z)H^2) \cap q(w)(H^2 \oplus \theta(z)H^2).$$

By [6], this shows that $[S_z, S^w_z] = 0$.

**Case 2.** Suppose that $\theta(0) \neq 0$. Let $\varphi'(z) = \varphi(z) - \langle \varphi, \theta \rangle \theta(z)$. Then $S_\varphi = S_{\varphi'}$, so that we may assume that $\varphi \perp \theta$. Write

$$\varphi(z) = \varphi_1(z) + \theta(z)z\varphi_2(z), \tag{4.13}$$

where $\varphi_1 \in H^2(\Gamma_z) \cap \theta H^2(\Gamma_z)$ and $\varphi_2 \in H^2(\Gamma_z)$. By (4.4), $\varphi_1(z) \neq 0$. Since $\theta(0) \neq 0$, $T^*_z\varphi_1(z) \neq 0$. For each $h \in N$, by (4.2) we can write

$$\varphi h = f_h + \theta g_h \in N \oplus \theta H^2.$$
Applying \( T^*_z \) for the both side of the above, we have
\[
\varphi T^*_z h + h(0, w)T^*_z \varphi = T^*_z f_h + g(0, w)T^*_z \theta + \theta T^*_z g_h.
\]
Hence by (4.12),
\[
\varphi T^*_z h = -h(0, w)T^*_z \varphi + T^*_z f_h + g_h(0, w)T^*_z \theta + \theta T^*_z g_h
\]
\[
= -h(0, w)T^*_z \varphi_1 + T^*_z f_h + g_h(0, w)T^*_z \theta + \theta(T^*_z g_h - h(0, w)\varphi_2).
\]
Note that
\[
-h(0, w)T^*_z \varphi_1 + T^*_z f_h + g_h(0, w)T^*_z \theta \perp \theta H^2.
\]
Since \( h \in N \), we have \( T^*_z h \in N \), so that by (4.2) we have
\[
-h(0, w)T^*_z \varphi_1 + T^*_z f_h + g_h(0, w)T^*_z \theta \in N.
\]
Since \( f_h \in N \), also we have \( T^*_z f_h \in N \) and
\[
(4.13) \quad -h(0, w)T^*_z \varphi_1 + g_h(0, w)T^*_z \theta \in N.
\]
Write
\[
\Theta(z) = \theta^2(z) - \theta(0)\theta(z).
\]
We have
\[
T^*_z = T^*_{\theta^2-\theta(0)\theta} = T^*_{\Theta}.
\]
Since
\[
T^*_z N \subset N,
\]
\[
-h(0, w)(T^*_z \varphi_1 + aT^*_z \theta) + g_h(0, w)T^*_z \theta \in N.
\]
Since \( \varphi_1 \in N \subset H^2 \circ \theta H^2 \), we have \( T^*_z \varphi_1 = 0 \). Since \( T^*_z \theta = -\theta(0) \), we get
\[
ah(0, w) - \theta(0)g(0, w) \in N.
\]
Since \( \theta(0) \neq 0 \),
\[
ah(0, w) - g(0, w) \in N.
\]
Thus we get
\[
ah(0, w) - g(0, w) \perp \theta(z)H^2.
\]
Because \( \theta(0) \neq 0 \), we have \( ah(0, w) - g(0, w) = 0 \). Hence by (4.9),
\[
h(0, w)T^*_z \varphi_1(z) \in N.
\]
Note that \( T^*_z \varphi_1(z) \neq 0 \). In the same way as Subcase 1.2,
\[
h(0, w)(H^2(\Gamma_z) \circ \theta(z)H^2(\Gamma_z)) \subset N \subset H^2 \circ \theta(z)H^2
\]
for every \( h \in N \). Since \( T^*_w N \subset N \) and \( N \neq \{0\} \), \( \{h(0, w) : h \in N\} \) is a nontrivial \( T^*_w \)-invariant subspace of \( H^2(\Gamma_w) \), so that
\[
\{h(0, w) : h \in N\} = H^2(\Gamma_w) \circ q(w)H^2(\Gamma_w)
\]
for either nontrivial inner function \( q(w) \) or \( q(w) = 0 \). Hence
\[
(H^2 \circ \theta(z)H^2) \circ q(w)(H^2 \circ \theta(z)H^2) \subset N.
\]
For every \( f \in N \), write
\[
f = \sum_{n=0}^{\infty} f_n(w)z^n.
\]
Since \( T_n^* N \subset N \), \( f_n(w) \in H^2(\Gamma w) \oplus q(w)H^2(\Gamma w) \) for every \( n \geq 0 \). Hence
\[
N \subset (H^2 \oplus \theta(z)H^2) \oplus q(w)(H^2 \oplus \theta(z)H^2).
\]
Therefore
\[
N = (H^2 \oplus \theta(z)H^2) \oplus q(w)(H^2 \oplus \theta(z)H^2).
\]
This shows that \([S_z, S_w^*] = 0\). This completes the proof. \( \square \)

References


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